

Group Actions on Monotone Skew-Product Semiflows with Applications

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Abstract

We discuss a general framework of monotone skew-product semiflows under a connected group action. In a prior work, a compact connected group G -action has been considered on a strongly monotone skew-product semiflow. Here we relax the requirement of strong monotonicity of the skew-product semiflows and the compactness of G , and establish a theory concerning symmetry or monotonicity properties of uniformly stable 1-cover minimal sets. We then apply this theory to show rotational symmetry of certain stable entire solutions for a class of non-autonomous

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reaction-diffusion equations on \mathbb{R}^n , as well as monotonicity of stable travelling waves of some nonlinear diffusion equations in time recurrent structures including almost periodicity and almost automorphy.

1 Introduction

In this article, we investigate monotone skew-product semiflows with certain symmetry such as ones with respect to rotation or translation. We will restrict our attention to solutions which are ‘stable’ in a certain sense and discuss the relation between stability and symmetry.

Historically, stability is in many cases known to imply some sort of symmetry. For autonomous (or time-periodic) parabolic equations, any stable equilibrium (or time-periodic) solution inherits the rotational symmetry of the domain Ω (see [3, 11] for bounded domain and [18, 19] for unbounded domain). In [18, 19], the symmetry of the stable solutions was also obtained for degenerate diffusion equations and systems of reaction-diffusion equations. Ni et al.[16] showed the spatially symmetric or monotonic structure of stable solutions in shadow systems as a limit of reaction-diffusion systems. It is now well known that parabolic equations and systems admitting the comparison principle define (strongly) monotone dynamical systems, whose concept was introduced in [8] (see [9, 24] for a comprehensive survey on the development of this theory). If the domain and the coefficients in such an equation or system exhibit a symmetry, then the dynamical system commutes with the action of some topological group G . Extensions and generalizations of group actions to a general framework of (strongly) monotone systems were given by [10, 13, 18, 19, 30].

Non-periodic and non-autonomous equations have been attracting more attention recently. A unified framework to study non-autonomous equations is based on the so-called skew-product semiflows (see [25, 26]). In [32], a compact connected group G -action was considered on a strongly monotone skew-product semiflow Π_t . Assuming that a minimal set K of Π_t is stable, it was proved in [32] that K is residually symmetric, and moreover, any uniformly stable orbit is asymptotically symmetric. In this article, motivated by Ogiwara and Matano [18, 19], we relax the restriction of strong monotonicity of the skew-product semiflow Π_t , as well as the compactness of the acting group G . To formulate our results precisely, we let K be a uniformly stable 1-cover of the base flow. Under the assumption that Π_t is only monotone and G is only connected, we establish the globally topological structure of the group orbit GK of K , where $GK = \{g \cdot (x, \omega) : g \in G \text{ and } (x, \omega) \in K\}$ (see Theorem B). Roughly speaking, the group orbit GK either coincides with K (which entails that K is G -symmetric); or otherwise, GK is a 1-dimensional continuous subbundle on the base, while each fibre of such bundle being totally ordered and homeomor-

phic to \mathbb{R} . In particular, when the second case holds, the uniform stability of K will imply the asymptotic uniform stability (see Theorem D).

Our main theorems are extensions of symmetry results in [18, 19] on stable equilibria (resp. fixed points) for continuous-time (resp. discrete-time) monotone systems. This enables us to investigate the symmetry of certain stable entire solutions of nonlinear reaction-diffusion equations in *time recurrent structures* (see Definition 2.6) on a symmetric domain. This is satisfied, for instance, when the reaction term is a *uniformly almost periodic* or, more generally, a *uniformly almost automorphic* function in t (see Section 2 for more details).

Since strong monotonicity of the skew-product semiflow is weakened, we are able to deal with the time-recurrent parabolic equation on an unbounded symmetric domain such as the entire space \mathbb{R}^n . For non-autonomous parabolic equations, radial symmetry has been shown to be a consequence of positivity of the solutions (see, e.g. [1, 7, 21, 22] and references therein). For non-autonomous parabolic equations on \mathbb{R}^n , we also refer to a series of very recent work by Poláčik [20, 21, 23] on this topic and its applications. In particular, he [21] proved that, under some symmetric conditions, any positive bounded entire solution decaying to zero at spatial infinity uniformly with respect to time is radially symmetric. However, as far as we know, symmetry properties of certain stable entire (possibly sign-changing) solutions of non-autonomous parabolic equations on \mathbb{R}^n have been hardly studied. By applying our abstract results mentioned above, we shall initiate our research on this aspect. More precisely, we show that (see Theorem 7.1) any uniformly stable entire solution is radially symmetric, provided that it satisfies certain module containment (see Definition 2.7) and decays to zero at spatial infinity uniformly with respect to time.

Note also that we have relaxed the requirement of compactness of the acting group G . This will allow one to discuss symmetry or monotonicity properties with respect to translation group. Based on this, one can investigate monotonicity of the uniformly-stable traveling waves for time-recurrent bistable reaction-diffusion equations or systems. Traveling waves in time-almost periodic nonlinear evolution equations governed by bistable nonlinearities were first established in a series of pioneer work by Shen [27]-[29]. In [27, 28], she proved the existence of such almost-periodic traveling waves, and showed that any such monotone traveling wave is uniformly-stable. By using our abstract results, on the other hand, we give a converse theorem (see Theorem 7.6) to that of Shen's, i.e., any uniformly-stable almost-periodic traveling wave is monotone. Moreover, we shall also show that any uniformly-stable almost-periodic traveling wave is uniformly stable with asymptotic phase (see Theorem 7.7). The same result as Theorem 7.7 can also be found in Shen [27]. But our approach (by Theorem D) was introduced in a very general framework, and hence, it can be applied in a rather general context and to wider classes of equations with little modification.

This paper is organized as follows. In section 2, we present some basic concepts and preliminary results in the theory of skew-product semiflows and almost periodic (automorphic) functions which will be important to our proofs. We state our main results in Section 3, where we also give standing assumptions characterizing our general framework. Sections 4-6 contain the proofs of our main results. In section 7, we apply our abstract theorems to obtain symmetry properties of certain stable entire (possibly sign-changing) solutions of non-autonomous parabolic equations on \mathbb{R}^n , as well as the monotonicity of stable almost-periodic traveling waves for time-recurrent reaction-diffusion equations.

2 Notation and preliminary results

In this section, we summarize some preliminary materials to be used in later sections. First, we summarize some lifting properties of compact dynamical systems. We then collect definitions and basic facts concerning monotone skew-product semiflows and order-preserving group actions. Finally, we give a brief review about uniformly almost periodic (automorphic) functions and flows.

Let Ω be a compact metric space with metric d_Ω , and $\sigma : \Omega \times \mathbb{R} \rightarrow \Omega$ be a continuous flow on Ω , denoted by (Ω, σ) or (Ω, \mathbb{R}) . As has become customary, we denote the value of σ at (ω, t) alternatively by $\sigma_t(\omega)$ or $\omega \cdot t$. By definition, $\sigma_0(\omega) = \omega$ and $\sigma_{t+s}(\omega) = \sigma_t(\sigma_s(\omega))$ for all $t, s \in \mathbb{R}$ and $\omega \in \Omega$. A subset $S \subset \Omega$ is *invariant* if $\sigma_t(S) = S$ for every $t \in \mathbb{R}$. A non-empty compact invariant set $S \subset \Omega$ is called *minimal* if it contains no non-empty, proper and invariant subset. We say that the continuous flow (Ω, \mathbb{R}) is *minimal* if Ω itself is a minimal set. Let (Z, \mathbb{R}) be another continuous flow. A continuous map $p : Z \rightarrow \Omega$ is called a *flow homomorphism* if $p(z \cdot t) = p(z) \cdot t$ for all $z \in Z$ and $t \in \mathbb{R}$. A flow homomorphism which is onto is called a *flow epimorphism* and a one-to-one flow epimorphism is referred as a *flow isomorphism*. We note that a homomorphism of minimal flows is already an epimorphism.

We say that a Banach space $(V, \|\cdot\|)$ is *ordered* if it contains a closed convex cone, that is, a non-empty closed subset $V_+ \subset V$ satisfying $V_+ + V_+ \subset V_+$, $\alpha V_+ \subset V_+$ for all $\alpha \geq 0$, and $V_+ \cap (-V_+) = \{0\}$. The cone V_+ induces an *ordering* on V via $x_1 \leq x_2$ if $x_2 - x_1 \in V_+$. We write $x_1 < x_2$ if $x_2 - x_1 \in V_+ \setminus \{0\}$. Given $x_1, x_2 \in V$, the set $[x_1, x_2] = \{x \in V : x_1 \leq x \leq x_2\}$ is called a *closed order interval* in V , and we write $(x_1, x_2) = \{x \in V : x_1 < x < x_2\}$.

A subset U of V is said to be *order convex* if for any $a, b \in U$ with $a < b$, the segment $\{a + s(b - a) : s \in [0, 1]\}$ is contained in U . And U is called *lower-bounded* (resp. *upper-bounded*) if there exists an element $a \in V$ such that $a \leq U$ (resp. $a \geq U$). Such an a is said to be a *lower bound* (resp. *upper bound*) for U . A lower bound a_0 is said to be the *greatest lower bound* (g.l.b.), if any other lower bound a satisfies $a \leq a_0$. Similarly, we can define the *least upper bound*

(l.u.b.).

Let $X = [a, b]_V$ with $a \ll b$ ($a, b \in V$) or $X = V_+$, or furthermore, X be a closed order convex subset of V . Throughout this paper, we always assume that, for any $u, v \in X$, the greatest lower bound of $\{u, v\}$, denoted by $u \wedge v$, exists and that $(u, v) \mapsto u \wedge v$ is a continuous mapping from $X \times X$ into X .

Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$. We consider a continuous *skew-product semiflow* $\Pi : \mathbb{R}^+ \times X \times \Omega \rightarrow X \times \Omega$ defined by

$$\Pi_t(x, \omega) = (u(t, x, \omega), \omega \cdot t), \quad \forall (t, x, \omega) \in \mathbb{R}^+ \times X \times \Omega, \quad (2.1)$$

satisfying (1) $\Pi_0 = \text{Id}$; (2) the *cocycle property*: $u(t+s, x, \omega) = u(s, u(t, x, \omega), \omega \cdot t)$, for each $(x, \omega) \in X \times \Omega$ and $s, t \in \mathbb{R}^+$. A subset $A \subset X \times \Omega$ is *positively invariant* if $\Pi_t(A) \subset A$ for all $t \in \mathbb{R}^+$; and *totally invariant* if $\Pi_t(A) = A$ for all $t \in \mathbb{R}^+$. The *forward orbit* of any $(x, \omega) \in X \times \Omega$ is defined by $O^+(x, \omega) = \{\Pi_t(x, \omega) : t \geq 0\}$, and the *omega-limit set* of (x, ω) is defined by $\mathcal{O}(x, \omega) = \{(\hat{x}, \hat{\omega}) \in X \times \Omega : \Pi_{t_n}(x, \omega) \rightarrow (\hat{x}, \hat{\omega}) \text{ } (n \rightarrow \infty) \text{ for some sequence } t_n \rightarrow \infty\}$. Clearly, if a forward orbit $O^+(x, \omega)$ is relatively compact, then the omega-limit set $\mathcal{O}(x, \omega)$ is a nonempty, compact and totally invariant subset in $X \times \Omega$ for Π_t .

Let $P : X \times \Omega \rightarrow \Omega$ be the natural projection. A compact positively invariant set $K \subset X \times \Omega$ is called a *1-cover* of the base flow if $P^{-1}(\omega) \cap K$ contains a unique element for every $\omega \in \Omega$. In this case, we denote the unique element of $P^{-1}(\omega) \cap K$ by $(c(\omega), \omega)$ and write $K = \{(c(\omega), \omega) : \omega \in \Omega\}$, where $c : \Omega \rightarrow X$ is continuous with

$$\Pi_t(c(\omega), \omega) = (c(\omega \cdot t), \omega \cdot t), \quad \forall t \geq 0,$$

and hence, $K \cap P^{-1}(\omega) = \{(c(\omega), \omega)\}$ for every $\omega \in \Omega$.

Next, we introduce some definition concerning the stability of the skew-product semiflow Π_t . A forward orbit $O^+(x_0, \omega_0)$ of Π_t is said to be *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $s \geq 0$ and $\|u(s, x_0, \omega_0) - x\| \leq \delta(\varepsilon)$ for certain $x \in X$, then for each $t \geq 0$, $\|u(t+s, x_0, \omega_0) - u(t, x, \omega_0 \cdot s)\| < \varepsilon$. The following definition is on the uniform stability for a compact positively invariant set $K \subset X \times \Omega$:

Definition 2.1 (Uniform stability for K). *A compact positively invariant set K is said to be uniformly stable if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$, called the modulus of uniform stability, such that, if $(x, \omega) \in K$, $(y, \omega) \in X \times \Omega$ are such that $\|x - y\| \leq \delta(\varepsilon)$, then*

$$\|u(t, x, \omega) - u(t, y, \omega)\| < \varepsilon \text{ for all } t \geq 0.$$

Remark 2.2. It is easy to be expected that all the trajectories in a uniformly stable set are uniformly stable. Conversely, if a trajectory has uniformly stable

property, its omega-limit set inherits it: that is, if $O^+(x_0, \omega_0)$ is relatively compact and uniformly stable, then the omega-limit set $\mathcal{O}(x_0, \omega_0)$ is a uniformly stable set with the same modulus of uniform stability as that of $O^+(x_0, \omega_0)$ (see [17, 25]).

The following Lemma is due to Novo et al [17, Proposition 3.6]:

Lemma 2.3. *Assume that (Ω, \mathbb{R}) is minimal. Let $O^+(x, \omega)$ be a forward orbit of Π_t which is relatively compact. If its omega-limit set $\mathcal{O}(x, \omega)$ contains a minimal set K which is uniformly stable, then $\mathcal{O}(x, \omega) = K$.*

For skew-product semiflows, we always use the order relation on each fiber $P^{-1}(\omega)$. We write $(x_1, \omega) \leq_\omega (x_2, \omega)$ ($<_\omega$) if $x_1 \leq x_2$ ($x_1 < x_2$). Without any confusion, we will drop the subscript “ ω ”. One can also define similar definitions and notations in $P^{-1}(\omega)$ as in X , such as order-intervals, the greatest lower bound, the least upper bound, etc.

Let A, B be two compact subsets of X . We define their Hausdorff metric

$$d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

where $d(x, B) = \inf_{y \in B} \|x - y\|$. We can also define the Hausdorff metric $d_{H, \omega}(A(\omega), B(\omega))$ for any two compact subset $A(\omega), B(\omega)$ of $P^{-1}(\omega)$. Again without any confusion, we drop the subscript “ ω ” and write $d_{H, \omega}(A(\omega), B(\omega))$ as $d_H(A(\omega), B(\omega))$ in the context.

Let K_1, K_2 be two positively invariant compact subsets of $X \times \Omega$. We write $K_1 \prec_r K_2$ if and only if for any $(x, \omega) \in K_1$, there exists some $(y, \omega) \in K_2$ such that $(x, \omega) <_r (y, \omega)$, and for any $(y, \omega) \in K_2$, there exists some $(x, \omega) \in K_1$ such that $(x, \omega) <_r (y, \omega)$, where \prec_r (resp. $<_r$) represents \preceq (resp. \leq) or \prec (resp. $<$). $K_1 \succ_r K_2$ is similarly defined. For such $K_1, K_2 \subset X \times \Omega$, the Hausdorff distance between K_1 and K_2 is defined as

$$d(K_1, K_2) = \sup_{\omega \in \Omega} d_H(K_1(\omega), K_2(\omega)),$$

where d_H is the Hausdorff metric for compact subsets in $P^{-1}(\omega)$.

Definition 2.4. The skew-product semiflow Π is *monotone* if

$$\Pi_t(x_1, \omega) \leq \Pi_t(x_2, \omega)$$

whenever $(x_1, \omega) \leq (x_2, \omega)$ and $t \geq 0$.

Let G be a metrizable topological group with unit element e . We say that G acts on the ordered space X if there exists a continuous mapping $\gamma : G \times X \rightarrow X$ such that $a \mapsto \gamma(a, \cdot)$ is a group homomorphism of G into $\text{Hom}(X)$, the group of homeomorphisms of X onto itself. For brevity, we write $\gamma(a, x) = ax$ for

$x \in X$ and identify the element $a \in G$ with its action $\gamma(a, \cdot)$. A group action γ is said to be *order-preserving* if, for each $a \in G$, the mapping $\gamma(a, \cdot) : X \rightarrow X$ is increasing, i.e. $x_1 \leq x_2$ in X implies $ax_1 \leq ax_2$. We say that γ *commutes* with the skew-product semiflow Π if

$$au(t, x, \omega) = u(t, ax, \omega), \text{ for any } (x, \omega) \in X \times \Omega, t \geq 0 \text{ and } a \in G.$$

For $x \in X$ the *group orbit* of x is the set $Gx = \{ax : a \in G\}$. A point $(x, \omega) \in X \times \Omega$ is said to be *symmetric* if $(Gx, \omega) = \{(x, \omega)\}$.

Due to the commutative property of G with Π_t , one has the following direct lemma:

Lemma 2.5. *For any $(x_0, \omega_0) \in X \times \Omega$ and $g \in G$, its omega-limit set $\mathcal{O}(x_0, \omega_0)$ satisfies*

$$g\mathcal{O}(x_0, \omega_0) = \mathcal{O}(gx_0, \omega_0),$$

where $g\mathcal{O}(x_0, \omega_0) = \{(gx, \omega) : (x, \omega) \in \mathcal{O}(x_0, \omega_0), \omega \in \Omega\}$.

Proof. Fix any $\omega \in \Omega$. Then for any $(x, \omega) \in \mathcal{O}(x_0, \omega_0)$, there exists a sequence $\{t_n\} \rightarrow \infty$ such that $\Pi_{t_n}(x_0, \omega_0) = (u(t_n, x_0, \omega_0), \omega_0 \cdot t_n) \rightarrow (x, \omega)$ as $n \rightarrow \infty$. So for any $g \in G$, we have $u(t_n, gx_0, \omega_0) = gu(t_n, x_0, \omega_0) \rightarrow gx$ as $n \rightarrow \infty$, and hence $(gx, \omega) \in \mathcal{O}(gx_0, \omega_0) \cap P^{-1}(\omega)$. Therefore, $g\mathcal{O}(x_0, \omega_0) \subset \mathcal{O}(gx_0, \omega_0)$.

Conversely, for any $(y, \omega) \in \mathcal{O}(gx_0, \omega_0)$, choose a sequence $\{s_n\} \rightarrow \infty$ such that $\Pi_{s_n}(gx_0, \omega_0) = (u(s_n, gx_0, \omega_0), \omega_0 \cdot s_n) \rightarrow (y, \omega)$ as $n \rightarrow \infty$. Thus, $gu(s_n, x_0, \omega_0) = u(s_n, gx_0, \omega_0) \rightarrow y$ as $n \rightarrow \infty$. Without loss of generality, we may assume that $u(s_n, x_0, \omega_0) \rightarrow x$ as $n \rightarrow \infty$. Therefore, $(x, \omega) \in \mathcal{O}(x_0, \omega_0)$ and $y = gx$, which implies that $(y, \omega) \in g\mathcal{O}(x_0, \omega_0)$. So we have proved $\mathcal{O}(gx_0, \omega_0) \subset g\mathcal{O}(x_0, \omega_0)$. By the arbitrariness of $\omega \in \Omega$, we directly derive the result. \square

We finish this section with the definitions of almost periodic (automorphic) functions and flows.

A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is *almost periodic* if, for any $\varepsilon > 0$, the set $T(\varepsilon) := \{\tau : |f(t+\tau) - f(t)| < \varepsilon, \forall t \in \mathbb{R}\}$ is relatively dense in \mathbb{R} . f is *almost automorphic* if for any $\{t'_n\} \subset \mathbb{R}$ there is a subsequence $\{t_n\}$ and a function $g : \mathbb{R} \rightarrow \mathbb{R}^n$ such that $f(t + t_n) \rightarrow g(t)$ and $g(t - t_n) \rightarrow f(t)$ hold pointwise.

Let D be a subset of \mathbb{R}^m . A continuous function $f : \mathbb{R} \times D \rightarrow \mathbb{R}^n; (t, u) \mapsto f(t, u)$, is said to be *admissible* if $f(t, u)$ is bounded and uniformly continuous on $\mathbb{R} \times K$ for any compact subset $K \subset D$. A function $f \in C(\mathbb{R} \times D, \mathbb{R}^n) (D \subset \mathbb{R}^m)$ is *uniformly almost periodic (automorphic) in t* , if f is both admissible and almost periodic (automorphic) in $t \in \mathbb{R}$.

Let $f \in C(\mathbb{R} \times D, \mathbb{R}^n) (D \subset \mathbb{R}^m)$ be admissible. Then $H(f) = \text{cl}\{f \cdot \tau : \tau \in \mathbb{R}\}$ is called the *hull of f* , where $f \cdot \tau(t, \cdot) = f(t + \tau, \cdot)$ and the closure is taken under the compact open topology. Moreover, $H(f)$ is compact and metrizable under

the compact open topology. The time translation $g \cdot t$ of $g \in H(f)$ induces a natural flow on $H(f)$.

Definition 2.6. An admissible function $f \in C(\mathbb{R} \times D, \mathbb{R}^n)$ is called *time recurrent* if $H(f)$ is minimal.

$H(f)$ is always minimal if f is uniformly almost periodic (automorphic) in t . Moreover, $H(f)$ is an almost periodic (automorphic) minimal flow when f is a uniformly almost periodic (automorphic) function in t (see, e.g. [25, 26]).

Let $f \in C(\mathbb{R} \times D, \mathbb{R}^n)$ be uniformly almost periodic (automorphic), and

$$f(t, x) \sim \sum_{\lambda \in \mathbb{R}} a_\lambda(x) e^{i\lambda t} \quad (2.2)$$

be a Fourier series of f (see [26, 31] for the definition and the existence of Fourier series). Then $\mathcal{S} = \{\lambda : a_\lambda(x) \not\equiv 0\}$ is called the Fourier spectrum of f associated to the Fourier series (2.2).

Definition 2.7. $\mathcal{M}(f) =$ the smallest additive subgroup of \mathbb{R} containing $\mathcal{S}(f)$ is called the frequency module of f .

Let $f, g \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be two uniformly almost periodic (automorphic) functions in t . We have the module containment $\mathcal{M}(f) \subset \mathcal{M}(g)$ if and only if there exists a flow epimorphism from $H(g)$ to $H(f)$ (see, [4] or [26, Section 1.3.4]). In particular, $\mathcal{M}(f) = \mathcal{M}(g)$ if and only if the flow $(H(g), \mathbb{R})$ is isomorphic to the flow $(H(f), \mathbb{R})$.

3 Main results

In this section our standing assumptions are as follows:

- (A1) Ω is minimal;
- (A2) G is a connected group acting on X in such a way that its action is order-preserving;
- (A3) G commutes with the monotone skew-product semiflow Π_t .

In what follows we will denote by K a minimal set of Π_t in $X \times \Omega$, which is a uniformly stable 1-cover of Ω . In the context, we also write $K = \{(\bar{u}_\omega, \omega) : \omega \in \Omega\}$, and $gK = \{(g\bar{u}_\omega, \omega) : \omega \in \Omega\}$ if an element $g \in G$ acts on K . The *group orbit of K* is defined as

$$GK = \{(g\bar{u}_\omega, \omega) \in X \times \Omega : g \in G \text{ and } \omega \in \Omega\}.$$

We will investigate the topological structure of GK in this paper.

For $\delta > 0$, we define a δ -neighborhood of K in $X \times \Omega$:

$$B_\delta(K) = \{(u, \omega) \in X \times \Omega : \|u - \bar{u}_\omega\| < \delta\}.$$

Hereafter, we impose the following additional condition on K :

(A4) There exists a $\delta > 0$ such that

- (i) the forward orbit $O^+(x_0, \omega_0)$ is relatively compact for any $(x_0, \omega_0) \in B_\delta(K)$; and moreover,
- (ii) if the ω -limit set $\mathcal{O}(x_0, \omega_0) \subset B_\delta(K)$ and $\mathcal{O}(x_0, \omega_0) \prec hK$ (resp. $\mathcal{O}(x_0, \omega_0) \succ hK$) for some $h \in G$, then there is a neighborhood $B(e) \subset G$ of e such that $\mathcal{O}(x_0, \omega_0) \prec ghK$ (resp. $\mathcal{O}(x_0, \omega_0) \succ ghK$) for any $g \in B(e)$.

Remark 3.1. In the case where Π_t is strongly monotone, (A4-ii) is automatically satisfied. Recall that Π_t is strongly monotone if $\Pi_t(x_1, \omega) \ll \Pi_t(x_2, \omega)$ whenever $(x_1, \omega) < (x_2, \omega)$ and $t > 0$ (see [26]). To derive (ii) of (A4), note that the total invariance of $\mathcal{O}(x_0, \omega_0)$ implies that, for any $(x, \omega) \in \mathcal{O}(x_0, \omega_0)$, there exists a neighborhood $B_{(x, \omega)}(e) \subset G$ of e such that $(x, \omega) \prec ghK$ for any $g \in B_{(x, \omega)}(e)$. Considering that $\mathcal{O}(x_0, \omega_0)$ is compact, one can find a neighborhood $B(e) \subset G$ such that $\mathcal{O}(x_0, \omega_0) \prec ghK$ for any $g \in B(e)$.

Remark 3.2. For continuous-time (discrete-time) monotone systems, assumption (A4) was first imposed by Ogiwara and Matano [18, 19] to investigate the monotonicity and convergence of the stable equilibria (fixed points). We here give a general version in non-autonomous cases. At first glance, one can observe that (A4) is just a local dynamical hypothesis nearby K . Accordingly, it should only yield a local total-ordering property of the group orbit GK nearby K (see Lemma A below). However, in what follows, we can see that it will surprisingly imply a globally topological characteristic of the whole group orbit GK (see Theorem B below), which is our main result in this paper.

Lemma A (Local ordering-property of GK nearby K). *Assume that (A1)-(A3) hold. Let K be a uniformly stable 1-cover of Ω and satisfies (A4). Then there exists a neighborhood $B(e) \subset G$ of e such that $gK \preceq K$ or $gK \succeq K$, for any $g \in B(e)$.*

Theorem B (Global topological structure of GK). *Assume that (A1)-(A3) hold and G is locally compact. Let K be a uniformly stable 1-cover of Ω and satisfies (A4). Then either of the following alternatives holds:*

- (i) $GK = K$, i.e., K is G -symmetric;
- (ii) There is a continuous bijective mapping $H : \Omega \times \mathbb{R} \rightarrow GK$ satisfying:
 - (a) For each $\alpha \in \mathbb{R}$, $H(\Omega, \alpha) = gK$ for some $g \in G$;
 - (b) For each $\omega \in \Omega$, $H(\omega, \mathbb{R}) = G\bar{u}_\omega$;
 - (c) H is strictly order-preserving with respect to $\alpha \in \mathbb{R}$, i.e.,

$$H(\omega, \alpha_1) \ll H(\omega, \alpha_2)$$

for any $\omega \in \Omega$ and any $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 < \alpha_2$.

Remark 3.3. Roughly speaking, Theorem B implies the following dichotomy: either K is G -symmetric; or otherwise, its group orbit GK is a 1-dimensional continuous subbundle on the base, while each fibre of such bundle being totally ordered and homeomorphic to \mathbb{R} .

Based on Theorem B, one can further deduce the following two useful theorems on symmetry of K , as well as its uniform stability with asymptotic phase.

Theorem C. *Assume all the hypotheses in Theorem B are satisfied. If G is a compact group, then K is G -symmetric.*

Theorem D (Uniform stability of K with asymptotic phase). *Assume all the hypotheses in Theorem B are satisfied. If $GK \neq K$, then there is a $\delta_* \in (0, \delta)$ such that, if $(u, \omega) \in B_{\delta_*}(K)$, then its ω -limit set $\mathcal{O}(u, \omega) = hK$ for some $h \in G$. Moreover,*

$$\|u(t, u, \omega) - h\bar{u}_{\omega \cdot t}\| \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

4 Globally topological structure of GK

In this section, we shall prove Theorems B and C under the assumption that the conclusion of Lemma A holds already. The proof of Lemma A will be given in Section 6. We first proceed to the following useful proposition.

Proposition 4.1. *For any $g \in G$, there exists a neighborhood $V_g \subset G$ of g such that $V_g K$ is totally-ordered, i.e.,*

$$g_1 K \preceq g_2 K \text{ or } g_1 K \succeq g_2 K, \quad \forall g_1, g_2 \in V_g.$$

Proof. Since the group G is metrizable, one can write $B(e)$ in Lemma A as $B(e) = \{g \in G : \rho(g, e) < \delta\}$ for some $\delta > 0$, where ρ denotes the right-invariant metric on G (cf. [15, Section 1.22]) satisfying $\rho(g\sigma, h\sigma) = \rho(g, h)$ for all $g, h, \sigma \in G$. Thus for any $g_1, g_2 \in G$, it follows from (A2) and Lemma A that

$$g_2 K \preceq g_1 K \text{ or } g_2 K \succeq g_1 K, \quad \text{whenever } \rho(g_1^{-1} g_2, e) < \delta. \quad (4.1)$$

Now for any $g \in G$, let $V_g = \{h \in G : \rho(g^{-1}, h^{-1}) < \frac{\delta}{2}\}$. It is not difficult to see that V_g is a neighborhood of g . Hence if $g_1, g_2 \in V_g$, then

$$\begin{aligned} \rho(g_1^{-1} g_2, e) &\leq \rho(g_1^{-1} g_2, g^{-1} g_2) + \rho(g^{-1} g_2, e) \\ &= \rho(g_1^{-1} g_2, g^{-1} g_2) + \rho(g^{-1} g_2, g_2^{-1} g_2) \\ &= \rho(g_1^{-1}, g^{-1}) + \rho(g^{-1}, g_2^{-1}) < \delta, \end{aligned}$$

because ρ is right-invariant. As a consequence, (4.1) implies that

$$g_1 K \preceq g_2 K \text{ or } g_1 K \succeq g_2 K, \forall g_1, g_2 \in V_g.$$

This completes the proof. \square

Now we are in position to prove our main result Theorem B:

Proof of Theorem B: For any two $g_1, g_2 \in G$, we write $g_1 \leq g_2$ whenever $g_1 K \preceq g_2 K$. Then a partial order “ \leq ” is induced in G . A subset $S \subset G$ is called totally-ordered if any two distinct elements of S are related.

We first claim that G is totally-ordered. To prove this, we define

$$\mathcal{F} = \{S \subset G : S \text{ is connected and totally-ordered}\}.$$

By virtue of Lemma A, $V_g \in \mathcal{F} \neq \emptyset$. Note that (\mathcal{F}, \subset) is a partially-ordered set. It follows from Zorn’s lemma that \mathcal{F} possesses a maximal element, say M . We first show that M is a closed subset of G . Consider the closure \bar{M} of M . Clearly, \bar{M} is connected. Now, for any $h_1, h_2 \in \bar{M}$, there exist sequences $\{g_n^1\}, \{g_n^2\} \subset M$ such that $g_n^1 \rightarrow h_1$, $g_n^2 \rightarrow h_2$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$, $g_n^1 \leq g_n^2$ or $g_n^1 \geq g_n^2$, because M is totally-ordered. By taking a subsequence $\{n_k\}$, if necessary, we obtain

$$g_{n_k}^1 \leq g_{n_k}^2, \forall k \in \mathbb{N} \text{ or } g_{n_k}^1 \geq g_{n_k}^2, \forall k \in \mathbb{N}.$$

Letting $k \rightarrow \infty$ in the above, one has $h_1 \leq h_2$ or $h_1 \geq h_2$, because the order “ \leq ” is closed. Hence \bar{M} is totally-ordered. By the maximality of M , we get $M = \bar{M}$, which implies that M is closed.

In order to show that M is also an open subset of G , we notice that for any $g \in M$, by Proposition 4.1, there is a neighborhood $V_g \subset G$ of g such that V_g is totally-ordered and connected. Suppose that M is not open. Then one can find some $g \in M$ and a sequence $\{g_n\}_{n=1}^\infty \subset V_g \setminus M$ such that $g_n \rightarrow g$ as $n \rightarrow \infty$. Since V_g is totally-ordered, we may also assume without loss of generality that $g_n > g$ for all $n \in \mathbb{N}$. Fix each $n \in \mathbb{N}$, we define

$$W_n^+ = \{h \in M \cap V_g : h \geq g_n\} \text{ and } W_n^- = \{h \in M \cap V_g : h \leq g_n\}.$$

A direct examination yields that (i) $M \cap V_g = W_n^+ \cup W_n^-$; (ii) $W_n^+ \cap W_n^- = \emptyset$ (Since $g_n \notin M$); (iii) $W_n^- \neq \emptyset$ (Since $g \in W_n^-$); and (iv) W_n^+, W_n^- are closed in $M \cap V_g$. By the connectivity of $M \cap V_g$, we have $W_n^+ = \emptyset$, and hence $W_n^- = M \cap V_g$. Since $g_n \notin M$, it entails that $M \cap V_g < g_n$ for each $n \in \mathbb{N}$. Letting $n \rightarrow \infty$, we therefore obtain

$$M \cap V_g \leq g. \tag{4.2}$$

Furthermore, we assert that $M \leq g$. Otherwise, noticing that $g \in M$ and M is totally-ordered, there is an $f \in M$ such that $f > g$. Since M is also connected

and locally compact, it follows from [18, Appendix, Proposition Y1, Page 434] that there is an order-preserving homeomorphism

$$\tilde{h} : [g, f]_M = \{h \in M : g \leq h \leq f\} \rightarrow [0, 1]$$

with $\tilde{h}(g) = 0$ and $\tilde{h}(f) = 1$. Thus by choosing $g_* \in \tilde{h}^{-1}(\delta)$ with $0 < \delta \ll 1$, one has $g_* \in (V_g \cap M) \setminus \{g\}$ and $g_* > g$, which is a contradiction to (4.2). Thus we have proved the assertion.

On the other hand, recall that $g_n \in V_g$ and $g_n > g$ for every $n \in \mathbb{N}$. Now we fix some g_n . Since V_g is connected, totally-ordered, and locally compact, [18, Appendix, Proposition Y1, Page 434] again implies that there is an order-preserving homeomorphism

$$\hat{h} : [g, g_n]_{V_g} = \{h \in V_g : g \leq h \leq g_n\} \rightarrow [0, 1]$$

with $\hat{h}(g) = 0$ and $\hat{h}(g_n) = 1$. Let $\hat{M} = M \cup [g, g_n]_{V_g}$. Then $\hat{M} \supsetneq M$. Due to the assertion in the above paragraph, we obtain that \hat{M} is connected and totally-ordered. This contradicts the maximality of M . Accordingly, M is an open subset of G .

Since M is both open and closed in G , it follows from the connectivity of G that $G = M$. Thus we have proved the claim that G is totally-ordered.

Based on this claim, precisely one of the following three alternatives must occur:

- (Alt_a) The least upper bound (l.u.b.) of G exists;
- (Alt_b) The greatest lower bound (g.l.b.) of G exists;
- (Alt_c) Neither l.u.b. nor g.l.b. of G exists.

If (Alt_a) holds, then one can find a $g_0 \in G$ such that

$$g\bar{u}_\omega \leq g_0\bar{u}_\omega \quad \text{for any } \omega \in \Omega \text{ and } g \in G.$$

In particular, $g_0^2\bar{u}_\omega \leq g_0\bar{u}_\omega$, and hence $g_0\bar{u}_\omega = g_0^{-1}(g_0^2\bar{u}_\omega) \leq g_0^{-1}(g_0\bar{u}_\omega) = \bar{u}_\omega \leq g_0\bar{u}_\omega$, which entails that $g_0\bar{u}_\omega = \bar{u}_\omega$ for any $\omega \in \Omega$. Consequently, $g^{-1}\bar{u}_\omega \leq \bar{u}_\omega$, and hence $\bar{u}_\omega = g(g^{-1}\bar{u}_\omega) \leq g\bar{u}_\omega \leq \bar{u}_\omega$, for any $g \in G$ and $\omega \in \Omega$. This implies that $GK = K$.

Similarly, one can obtain $GK = K$ provided that (Alt_b) is satisfied. Thus we have concluded the statement (i) of Theorem B.

Finally we assume that (Alt_c) holds. Then fix any $\omega \in \Omega$, $G\bar{u}_\omega$ is a connected, locally compact and totally ordered set in X . Moreover, $G\bar{u}_\omega$ has neither the l.u.b. nor the g.l.b. in X . It then follows from [18, Appendix, Proposition Y2, Page 434] that $G\bar{u}_\omega$ coincides with the image of a strictly order-preserving continuous path in X :

$$J_\omega : \mathbb{R} \rightarrow G\bar{u}_\omega \subset X. \tag{4.3}$$

Motivated by [2, Section 3], we choose an $\omega_0 \in \Omega$ and define the mapping

$$H : \Omega \times \mathbb{R} \rightarrow GK; (\omega, \alpha) \mapsto \mathcal{O}(J_{\omega_0}(\alpha)) \cap P^{-1}(\omega), \quad (4.4)$$

where J_{ω_0} comes from (4.3) with ω replaced by ω_0 . Then it is not hard to check (a)-(c) for H in the statement (ii) in Theorem B. We only need to show that H is a bijective continuous map.

To end this, we first note that H is surjective. Indeed, for any $(g\bar{u}_\omega, \omega) \in GK$, let the real number $\hat{\alpha} \in \mathbb{R}$ be such that $J_{\omega_0}(\hat{\alpha}) = g\bar{u}_{\omega_0}$. Then it is easy to see that $\mathcal{O}(J_{\omega_0}(\hat{\alpha})) \cap P^{-1}(\omega) = (g\bar{u}_\omega, \omega)$, because gK is a uniformly stable 1-cover of the base Ω . Consequently, $H(\omega, \hat{\alpha}) = (g\bar{u}_\omega, \omega)$, which implies that H is surjective.

Next we choose any $(\omega_i, \alpha_i) \in \Omega \times \mathbb{R}$, $i = 1, 2$, with $H(\omega_1, \alpha_1) = H(\omega_2, \alpha_2)$. For each α_i , there is a $g_i \in G$ such that $J_{\omega_0}(\alpha_i) = g_i\bar{u}_{\omega_0}$ for $i = 1, 2$. Again by the 1-cover property of g_iK ,

$$(g_1\bar{u}_{\omega_1}, \omega_1) = H(\omega_1, \alpha_1) = H(\omega_2, \alpha_2) = (g_2\bar{u}_{\omega_2}, \omega_2).$$

Combining with (4.3), we obtain that $\omega_1 = \omega_2$ and $g_1 = g_2$, which implies that $\alpha_1 = \alpha_2$. Thus H is injective.

In order to prove H is continuous, we choose any sequence $\{(\omega_k, \alpha_k)\}_{k=1}^\infty \subset \Omega \times \mathbb{R}$ with $(\omega_k, \alpha_k) \rightarrow (\omega_\infty, \alpha_\infty)$ as $k \rightarrow \infty$. Accordingly, for each $k = 1, 2, \dots, \infty$, we can find $g_k \in G$ such that $J_{\omega_0}(\alpha_k) = g_k\bar{u}_{\omega_0}$. Similarly as above, one can further obtain that

$$H(\omega_k, \alpha_k) = (g_k\bar{u}_{\omega_k}, \omega_k), \quad (4.5)$$

for $k = 1, 2, \dots, \infty$. Since $\alpha_k \rightarrow \alpha_\infty$, we have $g_k\bar{u}_{\omega_0} \rightarrow g_\infty\bar{u}_{\omega_0}$ as $k \rightarrow \infty$. Note also that $g_\infty K$ is uniformly stable. Then for any $\varepsilon > 0$, there exists an integer $N = N(\varepsilon) > 0$ such that $\|u(t, g_k\bar{u}_{\omega_0}, \omega_0) - u(t, g_\infty\bar{u}_{\omega_0}, \omega_0)\| \leq \varepsilon/3$ for all $k \geq N$ and $t \geq 0$. By letting $t \rightarrow \infty$, it yields that, if $k \geq N$ then

$$\|g_k\bar{u}_\omega - g_\infty\bar{u}_\omega\| \leq \varepsilon/3, \quad (4.6)$$

uniformly for all $\omega \in \Omega$. Moreover, for such ε and N (choose N larger if necessary), it is easy to see that

$$\|\omega_k - \omega_\infty\| < \varepsilon/3 \quad \text{and} \quad \|g_\infty\bar{u}_{\omega_k} - g_\infty\bar{u}_{\omega_\infty}\| < \varepsilon/3, \quad (4.7)$$

for all $k \geq N$. By virtue of (4.5)-(4.7), we have

$$\begin{aligned} \|H(\omega_k, \alpha_k) - H(\omega_\infty, \alpha_\infty)\| &= \|(g_k\bar{u}_{\omega_k}, \omega_k) - (g_\infty\bar{u}_{\omega_\infty}, \omega_\infty)\| \\ &\leq \|\omega_k - \omega_\infty\| + \|g_k\bar{u}_{\omega_k} - g_\infty\bar{u}_{\omega_k}\| + \|g_\infty\bar{u}_{\omega_k} - g_\infty\bar{u}_{\omega_\infty}\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

for all $k \geq N$. We have proved that H is continuous. \square

Proof of Theorem C. Since G is compact, both (Alt_a) and (Alt_b) are satisfied. Then we directly deduce that $GK = K$ from the proof above. \square

5 Uniformly stability of K with asymptotic phase

In this section, we will prove the asymptotic phase of the uniformly stable minimal set K , i.e., Theorem D in Section 3. We first present the following useful lemma:

Lemma 5.1. *Assume all the hypotheses in Theorem B are satisfied. Assume also that $GK \neq K$. Then there exists a $\delta_0 > 0$ such that, if $(u, \omega) \in B_{\delta_0}(K)$ satisfies $\mathcal{O}(u, \omega) \preceq g_1 K$ for some $g_1 \in G$, then $\mathcal{O}(u, \omega) = g_2 K$ for some $g_2 \in G$. The same conclusion also holds if $(u, \omega) \in B_{\delta_0}(K)$ satisfies $\mathcal{O}(u, \omega) \succeq g_1 K$.*

Proof. Without loss of generality, we only prove the first statement. Suppose that there exists a sequence $\{(u_m, \omega_m)\}_{m=1}^\infty \subset X \times \Omega$ such that, for each $m \geq 1$,

- (i) $(u_m, \omega_m) \in B_{1/m}(K)$;
- (ii) $\mathcal{O}(u_m, \omega_m) \preceq g_m^1 K$, for some $g_m^1 \in G$; and
- (iii) $\mathcal{O}(u_m, \omega_m) \neq gK$, for any $g \in G$.

By virtue of Lemma 2.3, (iii) implies that

$$gK \not\preceq \mathcal{O}(u_m, \omega_m) \quad \text{for all } m \geq 1 \text{ and } g \in G. \quad (5.1)$$

Now we claim that

$$d(\mathcal{O}(u_m, \omega_m), K) \rightarrow 0, \quad \text{as } m \rightarrow \infty. \quad (5.2)$$

In fact, since K is uniformly stable, for any $\varepsilon > 0$ there exists a $\tilde{\delta}(\varepsilon)$ such that, if $\|(y, \omega) - (\bar{u}_\omega, \omega)\| < \tilde{\delta}(\varepsilon)$ then $\|u(t, y, \omega) - u(t, \bar{u}_\omega, \omega)\| < \varepsilon$ for all $t \geq 0$. Then, for $(u_m, \omega_m) \in B_{1/m}(K)$ with m sufficiently large, one has $\|(u_m, \omega_m) - (\bar{u}_{\omega_m}, \omega_m)\| < \frac{1}{m} < \tilde{\delta}(\varepsilon)$, and hence, $\|u(t, u_m, \omega_m) - u(t, \bar{u}_{\omega_m}, \omega_m)\| < \varepsilon$ for all $t \geq 0$. By the minimality of Ω , it then follows that $\|(y, \omega) - (\bar{u}_\omega, \omega)\| \leq \varepsilon$ whenever $(y, \omega) \in \mathcal{O}(u_m, \omega_m)$. Thus we have proved the claim.

Now fix $m \in \mathbb{N}$. We define $A_m = \{g \in G : \mathcal{O}(u_m, \omega_m) \preceq gK\}$. Clearly, A_m is nonempty (because $g_m^1 \in A_m$ by (ii)) and closed in G . By virtue of (5.1) and (5.2), one obtains that $A_m = \{g \in G : \mathcal{O}(u_m, \omega_m) \prec gK\}$, and moreover, $\mathcal{O}(u_m, \omega_m) \subset B_\delta(K)$ as long as m is sufficiently large. Here the δ is adopted from condition (A4) in Section 3.

As a consequence, (A4) entails that A_m is also open for all m sufficiently large. Since G is connected, $A_m = G$ for all m sufficiently large. This then implies that

$$\mathcal{O}(u_m, \omega_m) \preceq gK, \quad \forall g \in G,$$

for all m sufficiently large. By letting $m \rightarrow \infty$ in the above inequality, (5.2) yields that $K \preceq gK$, $\forall g \in G$. Replacing g with g^{-1} and applying g on both sides, we get $gK \preceq K$. Hence $gK = K$ for all $g \in G$, a contraction. We have completed the proof of the lemma. \square

Proof of Theorem D. Let $\delta_0 > 0$ be defined in Lemma 5.1. We take a $\delta_* \in (0, \min\{\delta, \delta_0\})$ such that $(u \wedge \bar{u}_\omega, \omega) \in B_{\delta_0}(K)$ whenever $(u, \omega) \in B_{\delta_*}(K)$. Since $u \wedge \bar{u}_\omega \leq \bar{u}_\omega$, one has $\mathcal{O}(u \wedge \bar{u}_\omega, \omega) \preceq K$. It then follows from Lemma 5.1 that $\mathcal{O}(u \wedge \bar{u}_\omega, \omega) = g_* K$ for some $g_* \in G$. Note also that $u \wedge \bar{u}_\omega \leq u$. Then $g_* K \preceq \mathcal{O}(u, \omega)$. Applying Lemma 5.1 again, we obtain that $\mathcal{O}(u, \omega) = gK$ for some $g \in G$. This completes the proof. \square

6 Proof of Lemma A

Proof of Lemma A. First we shall show that *there exists a neighborhood $B(e) \subset G$ of e such that for any $g \in B(e)$, one has $g\bar{u}_{\omega_0} \leq \bar{u}_{\omega_0}$ or $g\bar{u}_{\omega_0} \geq \bar{u}_{\omega_0}$ for some $\omega_0 \in \Omega$* . Otherwise, one can find a sequence $\{g_n\}_{n=0}^\infty \subset G$ with $g_n \rightarrow e$ as $n \rightarrow \infty$ such that

$$g_n \bar{u}_\omega \not\leq \bar{u}_\omega \text{ and } g_n \bar{u}_\omega \not\geq \bar{u}_\omega, \text{ for all } n \geq 0 \text{ and } \omega \in \Omega. \quad (6.1)$$

In what follows, we will deduce a contradiction from (6.1). For this purpose, we fix an $\omega_0 \in \Omega$, and due to (A4-i), we define $K_n = \mathcal{O}(g_n \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0)$ for all n sufficiently large. Without loss of generality, one may also assume that K_n is defined for all $n \in \mathbb{N}$. Clearly, $K = \mathcal{O}(\bar{u}_{\omega_0}, \omega_0)$. Then one can obtain the following three facts, the proof of which will be presented in the end of this section (see Propositions 6.1-6.3):

(F1) $K_n \prec K$ and $K_n \prec g_n K$ for all $n \in \mathbb{N}$.

(F2) $d(K_n, K) \rightarrow 0$, as $n \rightarrow \infty$.

(F3) Given the $\delta > 0$ in (A4), there exists a neighborhood $\hat{B}(e) \subset G$ of e and $N_0 \in \mathbb{N}$ such that

$$d(gK_n, K) \leq \delta \text{ and } d(g_n^{-1}gK_n, K) \leq \delta,$$

for all $g \in \hat{B}(e)$ and $n \geq N_0$.

For such $\hat{B}(e)$ and $N_0 \in \mathbb{N}$ in (F3), we take a neighborhood $B(e) \subset G$ of e with $B(e) \subset \overline{\hat{B}(e)} \subset \hat{B}(e)$, and define

$$A_n = \{g \in \overline{B(e)} : gK_n \preceq K \text{ and } g_n^{-1}gK_n \preceq K\}$$

for each $n \geq N_0$. By (F1), it is easy to see that $e \in A_n \neq \emptyset$. Moreover, A_n is closed in $\overline{B(e)}$. We assert that

$$A_n = \{g \in \overline{B(e)} : gK_n \prec K \text{ and } g_n^{-1}gK_n \prec K\}. \quad (6.2)$$

Indeed, for $g \in A_n$, suppose that there exists some $(y, \tilde{\omega}) \in K_n$ such that $gy = \bar{u}_{\tilde{\omega}}$. Then by $g_n^{-1}gK_n \preceq K$ we have $g_n^{-1}gy \leq \bar{u}_{\tilde{\omega}}$. It entails that $g_n^{-1}\bar{u}_{\tilde{\omega}} \leq \bar{u}_{\tilde{\omega}}$, and hence $\bar{u}_{\tilde{\omega}} \leq g_n \bar{u}_{\tilde{\omega}}$, contradicting to (6.1). Similarly, for such $g \in A_n$, suppose that there exists $(z, \hat{\omega}) \in K_n$ such that $g_n^{-1}gz = \bar{u}_{\hat{\omega}}$. Then by $gK_n \preceq K$

we have $gz \leq \bar{u}_{\hat{\omega}}$, which yields $g_n^{-1}\bar{u}_{\hat{\omega}} \geq g_n^{-1}gz = \bar{u}_{\hat{\omega}}$, and hence $\bar{u}_{\hat{\omega}} \geq g_n\bar{u}_{\hat{\omega}}$, contradicting to (6.1) again. So we have proved the assertion (6.2).

Now fix $n \geq N_0$ and let $g \in A_n$, we write $v_{g,n}^0 := g(g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0})$ and $w_{g,n}^0 := g_n^{-1}g(g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0})$. Then by (F3) and Lemma 2.5, one obtains that

$$d(\mathcal{O}(v_{g,n}^0, \omega_0), K) \leq \delta \quad \text{with} \quad \mathcal{O}(v_{g,n}^0, \omega_0) = gK_n \prec K,$$

and

$$d(\mathcal{O}(w_{g,n}^0, \omega_0), K) \leq \delta \quad \text{with} \quad \mathcal{O}(w_{g,n}^0, \omega_0) = g_n^{-1}gK_n \prec K.$$

Accordingly, the condition (A4) implies that there exist neighborhoods $B_1(e)$, $B_2(e) \subset G$ of e such that $gK_n = \mathcal{O}(v_{g,n}^0, \omega_0) \prec B_1(e)K$ and $g_n^{-1}gK_n = \mathcal{O}(w_{g,n}^0, \omega_0) \prec B_2(e)K$, where $B_i(e)K = \{gK : g \in B_i(e)\}$ for $i = 1, 2$. As a consequence,

$$(B_1(e))^{-1}gK_n \prec K \quad \text{and} \quad (B_2(e))^{-1}g_n^{-1}gK_n \prec K. \quad (6.3)$$

Clearly, $(B_1(e))^{-1}g$ and $(B_2(e))^{-1}g_n^{-1}g$ are neighborhoods of g and $g_n^{-1}g$, respectively. Moreover, by the continuity of $g \mapsto g_n^{-1}g$, one can find a neighborhood V_g of g in G , such that $g_n^{-1}V_g \subset (B_2(e))^{-1}g_n^{-1}g$. Thus by (6.3) we have $g_n^{-1}V_gK_n \prec K$. Now let $W_g := \overline{B(e)} \cap V_g \cap (B_1(e))^{-1}g$. Then by (6.3) again, W_g is a neighborhood of g in $\overline{B(e)}$ satisfying

$$W_gK_n \prec K \quad \text{and} \quad g_n^{-1}W_gK_n \prec K.$$

Therefore, $W_g \subset A_n$, which implies that A_n is also open in $\overline{B(e)}$. Thus by the connectivity of G (and hence the connectivity of $\overline{B(e)}$), one has

$$A_n = \overline{B(e)}, \quad \forall n \geq N_0.$$

Consequently,

$$\overline{B(e)}K_n \preceq K \quad \text{and} \quad g_n^{-1}\overline{B(e)}K_n \preceq K$$

for all $n \geq N_0$. Letting $n \rightarrow \infty$ in the above, by (F2), we then have

$$\overline{B(e)}K \preceq K. \quad (6.4)$$

Since $g_n \rightarrow e$ as $n \rightarrow \infty$, (6.4) implies that $g_n\bar{u}_{\omega} \leq \bar{u}_{\omega}$ for all $\omega \in \Omega$ and n sufficiently large, which is a contradiction to (6.1).

Therefore, we have proved that there exists a neighborhood $B(e) \subset G$ of e such that for any $g \in B(e)$, one has $g\bar{u}_{\omega_0} \leq \bar{u}_{\omega_0}$ or $g\bar{u}_{\omega_0} \geq \bar{u}_{\omega_0}$ for some $\omega_0 \in \Omega$.

Without loss of generality, we assume that $g\bar{u}_{\omega_0} \leq \bar{u}_{\omega_0}$. Then the monotonicity of Π_t implies $g\bar{u}_{\omega_0 \cdot t} \leq \bar{u}_{\omega_0 \cdot t}$ for any $t \geq 0$. Now for any $\omega \in \Omega$, we choose a sequence $\{t_n\} \rightarrow \infty$ such that $\omega_0 \cdot t_n \rightarrow \omega$ as $n \rightarrow \infty$. By the 1-cover property of K , one has $\bar{u}_{\omega_0 \cdot t_n} \rightarrow \bar{u}_{\omega}$ as $n \rightarrow \infty$. Thus, by letting $n \rightarrow \infty$, we obtain that $g\bar{u}_{\omega} \leq \bar{u}_{\omega}$ for any $\omega \in \Omega$. This implies that $gK \preceq K$ for any

$g \in B(e)$. Similarly, one can obtain that $K \preceq gK$ for any $g \in B(e)$ provided that $\bar{u}_{\omega_0} \leq g\bar{u}_{\omega_0}$. Accordingly, we conclude that for $K = \{(\bar{u}_\omega, \omega) : \omega \in \Omega\}$, there holds

$$gK \preceq K \text{ or } gK \succeq K, \forall g \in B(e).$$

This is the exact statement of Lemma A. \square

Finally, it only left to check (F1)-(F3) above. This will be done in the following three propositions.

Proposition 6.1. (F1) holds, i.e., $K_n \prec K$ and $K_n \prec g_n K$ for all $n \in \mathbb{N}$.

Proof. Note that $g_n \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0} < \bar{u}_{\omega_0}$ (resp. $< g_n \bar{u}_{\omega_0}$). It then follows from the monotonicity of Π_t that

$$\Pi_t(g_n \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \leq \Pi_t(\bar{u}_{\omega_0}, \omega_0) \text{ (resp. } \leq \Pi_t(g_n \bar{u}_{\omega_0}, \omega_0)), \quad (6.5)$$

for all $t \geq 0$. So, for any $(x, \omega) \in K_n$, one can find a sequence $\{t_k\} \rightarrow \infty$ ($k \rightarrow \infty$) such that $\Pi_{t_k}(g_n \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \rightarrow (x, \omega)$ as $k \rightarrow \infty$. Since K is a 1-cover, one has $\Pi_{t_k}(\bar{u}_{\omega_0}, \omega_0) \rightarrow (\bar{u}_\omega, \omega)$. Then (6.5) implies that $(x, \omega) \leq (\bar{u}_\omega, \omega)$. As a consequence, $K_n \preceq K$. Similarly, we can also obtain $K_n \preceq g_n K$ for every $n \in \mathbb{N}$.

Now we claim that $K_n \prec K$ (resp. $\prec g_n K$) for all $n \in \mathbb{N}$. Otherwise, there exist some $N \in \mathbb{N}$ and $(x, \tilde{\omega}) \in K_N$ such that

$$(x, \tilde{\omega}) = (\bar{u}_{\tilde{\omega}}, \tilde{\omega}) \text{ (resp. } (= g_N \bar{u}_{\tilde{\omega}}, \tilde{\omega})). \quad (6.6)$$

Choose a sequence $\{s_k\} \rightarrow \infty$ ($k \rightarrow \infty$) such that $\Pi_{s_k}(g_N \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \rightarrow (x, \tilde{\omega})$ as $k \rightarrow \infty$. Since

$$\begin{aligned} \Pi_t(g_n \bar{u}_\omega \wedge \bar{u}_\omega, \omega) &\leq \Pi_t(g_n \bar{u}_\omega, \omega) \wedge \Pi_t(\bar{u}_\omega, \omega) \\ &= (g_n \bar{u}_{\omega \cdot t}, \omega \cdot t) \wedge (\bar{u}_{\omega \cdot t}, \omega \cdot t) = (g_n \bar{u}_{\omega \cdot t} \wedge \bar{u}_{\omega \cdot t}, \omega \cdot t) \end{aligned}$$

for all $\omega \in \Omega$, $t \geq 0$ and $n \in \mathbb{N}$, it follows that

$$\Pi_{s_k}(g_N \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \leq (g_N \bar{u}_{\omega_0 \cdot s_k} \wedge \bar{u}_{\omega_0 \cdot s_k}, \omega_0 \cdot s_k).$$

Letting $k \rightarrow \infty$ in the above, by the continuity of \bar{u}_ω w.r.t. $\omega \in \Omega$, we then get

$$(x, \tilde{\omega}) \leq (g_N \bar{u}_{\tilde{\omega}} \wedge \bar{u}_{\tilde{\omega}}, \tilde{\omega}) < (\bar{u}_{\tilde{\omega}}, \tilde{\omega}) \text{ (resp. } (< g_N \bar{u}_{\tilde{\omega}}, \tilde{\omega})),$$

where the last inequality is from (6.1). Accordingly, a contradiction to (6.6) is obtained. Thus we have proved $K_n \prec K$ (resp. $\prec g_n K$) for all $n \in \mathbb{N}$. \square

Proposition 6.2. (F2) holds, i.e., $d(K_n, K) \rightarrow 0$, as $n \rightarrow \infty$.

Proof. Note that $g_n \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0} \rightarrow \bar{u}_{\omega_0}$ as $n \rightarrow \infty$. Since K is a uniformly stable 1-cover of Ω , it entails that, for any $\varepsilon > 0$, there is some $N_1 \in \mathbb{N}$ such that

$$\|u(t, g_n \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) - \bar{u}_{\omega_0, t}\| < \varepsilon \quad (6.7)$$

for all $n \geq N_1$ and $t \geq 0$. Choose any $(x, \omega) \in K_n$, there exists a sequence $\{t_k\} \rightarrow \infty$ ($k \rightarrow \infty$) such that $\Pi_{t_k}(g_n \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \rightarrow (x, \omega)$ as $k \rightarrow \infty$. By taking a subsequence, if necessary, we get that $\Pi_{t_k}(\bar{u}_{\omega_0}, \omega_0) \rightarrow (\bar{u}_\omega, \omega)$ as $k \rightarrow \infty$. Hence by (6.7), we have that $\|x - \bar{u}_\omega\| \leq \varepsilon$ for all $(x, \omega) \in K_n$ and $n \geq N_1$. Recall that $d(K_n, K) = \sup_{(x, \omega) \in K_n} \|x - \bar{u}_\omega\|$. Consequently, $d(K_n, K) \leq \varepsilon$ for all $n \geq N_1$, which implies that $d(K_n, K) \rightarrow 0$ as $n \rightarrow \infty$. \square

Proposition 6.3. (F3) holds, i.e., for the $\delta > 0$ in (A4), there exists a neighborhood $\hat{B}(e) \subset G$ of e and $N_0 \in \mathbb{N}$ such that

$$d(gK_n, K) \leq \delta \quad \text{and} \quad d(g_n^{-1}gK_n, K) \leq \delta,$$

for all $g \in \hat{B}(e)$ and $n \geq N_0$.

Proof. Firstly, suppose that there exist a sequence $\{\tilde{g}_n\}_{n=0}^\infty \subset G$ with $\tilde{g}_n \rightarrow e$ and a subsequence of $\{K_n\}_{n=0}^\infty$, still denoted by $\{K_n\}_{n=0}^\infty$, such that

$$d(\tilde{g}_n K_n, K) = \sup_{(y, \omega) \in K_n} \|\tilde{g}_n y - \bar{u}_\omega\| > \delta$$

for all $n \in \mathbb{N}$. Then one can choose some $(y_n, \omega_n) \in K_n$ such that

$$\|\tilde{g}_n y_n - \bar{u}_{\omega_n}\| > \delta. \quad (6.8)$$

Without loss of generality we assume that $\omega_n \rightarrow \omega$ in Ω as $n \rightarrow \infty$. Now we claim that $y_n \rightarrow \bar{u}_\omega$ as $n \rightarrow \infty$. Indeed, Proposition 6.2 suggests that, for any $\varepsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that

$$\|z - \bar{u}_\omega\| < \varepsilon, \quad \text{for all } (z, \omega) \in K_n \text{ and } n > N.$$

So $\|y_n - \bar{u}_{\omega_n}\| < \varepsilon$ for all $n > N$, because $(y_n, \omega_n) \in K_n$. Due to the continuity of \bar{u}_ω w.r.t. $\omega \in \Omega$, one has

$$\|y_n - \bar{u}_\omega\| \leq \|y_n - \bar{u}_{\omega_n}\| + \|\bar{u}_{\omega_n} - \bar{u}_\omega\| < \varepsilon + \varepsilon = 2\varepsilon, \quad \forall n > \bar{N}$$

for some positive integer $\bar{N} > N$. Thus, we have proved the claim. Then by letting $n \rightarrow \infty$ in (6.8), we obtain $\|\bar{u}_\omega - \bar{u}_\omega\| = \|e\bar{u}_\omega - \bar{u}_\omega\| \geq \delta$, a contradiction. Such contradiction implies that one can find a neighborhood $B_1(e)$ of e and some $N_1 \in \mathbb{N}$ such that $d(gK_n, K) \leq \delta$ for all $g \in B_1(e)$ and $n \geq N_1$.

Secondly, suppose that there exist a sequence $\{h_n\}_{n=0}^\infty \subset G$ with $h_n \rightarrow e$ and a subsequence $\{K_{j_n}\}_{n=0}^\infty$ of $\{K_n\}_{n=0}^\infty$ such that

$$d(g_{j_n}^{-1}h_n K_{j_n}, K) = \sup_{(y, \omega) \in K_{j_n}} \|g_{j_n}^{-1}h_n y - \bar{u}_\omega\| > \delta \quad \text{for all } n \in \mathbb{N}.$$

Then there exists some $(y_{j_n}, \omega_{j_n}) \in K_{j_n}$ such that $\|g_{j_n}^{-1}h_n y_{j_n} - \bar{u}_{\omega_{j_n}}\| > \delta$. Noticing $g_{j_n}^{-1} \rightarrow e$, one can repeat the same argument above to deduce a contradiction. Thus, again one can find a neighborhood $B_2(e)$ of e and some $N_2 \in \mathbb{N}$ such that $d(g_n^{-1}gK_n, K) \leq \delta$ for all $g \in B_2(e)$ and $n \geq N_2$.

Finally, let $\hat{B}(e) = B_1(e) \cap B_2(e)$ and $N_0 = \max\{N_1, N_2\}$. We have completed the proof of (F3). \square

7 Applications to parabolic equations

In this section we give some examples of second order parabolic equations in time-recurrent structures which generate monotone skew-product semiflows satisfying (A1)-(A4).

7.1 Rotational symmetry

Assume that $\Omega \subset \mathbb{R}^n$ is a (possibly unbounded) rotationally symmetric domain with smooth boundary $\partial\Omega$. Let G be a connected closed subgroup of the rotation group $SO(n)$. Ω is called G -symmetric if it is G -invariant in the sense that $gx \in \Omega$ whenever $x \in \Omega$ and $g \in G$. A typical example of such a bounded domain is a ball, a spherical shell, a solid torus or any other body of rotation. While, typical unbounded domains include cylindrical domain or \mathbb{R}^n itself. In [32], asymptotic symmetry has been investigated for the bounded domains. In this section, we focus on unbounded domains and, for brevity, we will present the following example on \mathbb{R}^n . As a matter of fact, general unbounded G -symmetric domains can be dealt with as well.

Consider the following initial value problem on \mathbb{R}^n :

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + f(t, x, u), & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (7.1)$$

Here the nonlinearity $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be a C^1 -admissible (with $D = \mathbb{R}^{n+1}$) and uniformly almost periodic in t , real-valued function.

In what follows we assume that

(f 1) $f(t, gx, u) = f(t, x, u)$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $g \in G$ and $t \in \mathbb{R}$;

(f 2) $f(t, x, 0) = 0$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$;

(f 3) there exist positive numbers ϵ_0, R_0, α such that $\frac{\partial f}{\partial u}(t, x, u) \leq -\alpha$ for all $|x| \geq R_0$, $|u| \leq \epsilon_0$ and $t \in \mathbb{R}$.

Let X be defined by

$$C_{\text{unif}}(\mathbb{R}^n) = \{u(x) : u \text{ is bounded and uniformly continuous on } \mathbb{R}^n\}$$

with the L^∞ -topology. Let $Y = H(f)$ be the hull of the nonlinearity f . Then, for any $g \in Y$, the function g is uniformly almost periodic in t and satisfies all the above assumptions (f 1)-(f 3). As a consequence, (7.1) gives rise to a family of equations associated to each $g \in Y$:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g(t, x, u), & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \quad (7.1_g)$$

By standard theory for parabolic equations (see [5, 6]), for every $u_0 \in X$ and $g \in H(f)$, equation (7.1_g) admits a (locally) unique classical solution $u(t, \cdot, u_0, g)$ in X with $u(0, \cdot, u_0, g) = u_0$. This solution also continuously depends on $g \in Y$ and $u_0 \in X$ (see, e.g. [6, 14]). Therefore, (7.1_g) defines a (local) skew-product semiflow Π_t on $X \times Y$ with

$$\Pi_t(u_0, g) = (u(t, \cdot, u_0, g), g \cdot t), \quad \forall (u_0, g) \in X \times Y, t \geq 0.$$

We define an order relation in X by

$$u \leq v \text{ if } u(x) \leq v(x), \quad \forall x \in \mathbb{R}^n.$$

The action of G on \mathbb{R}^n induces a group action on X by

$$a : u(x) \mapsto u(a^{-1}x).$$

Clearly, (A1)-(A3) in Section 3 are fulfilled.

Theorem 7.1 (Rotational symmetry). *Any uniformly L^∞ -stable entire (possibly sign-changing) solution $\bar{u}_f(t, x)$ of (7.1) (with $\mathcal{M}(\bar{u}_f) \subset \mathcal{M}(f)$) satisfying*

$$\sup_{t \in \mathbb{R}} |\bar{u}_f(t, x)| \rightarrow 0, \quad \text{as } |x| \rightarrow \infty \quad (7.2)$$

is G -symmetric, i.e., $\bar{u}_f(t, gx) = \bar{u}_f(t, x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $g \in G$.

For the entire solution $\bar{u}_f(t, x)$ given in Theorem 7.1, clearly, $E := \text{cl}\{\bar{u}_f(t, \cdot) \in X : t \in \mathbb{R}\}$ is a 1-cover of $H(f)$, because \bar{u}_f is uniformly stable. Thus one can write $E = \{\bar{u}_g(0, \cdot) \in X : g \in H(f)\}$ with $\bar{u}_f(t, \cdot) = \bar{u}_{f \cdot t}(0, \cdot)$ for all $t \in \mathbb{R}$. Let $K := \{(\bar{u}_g(0, \cdot), g) : g \in H(f)\}$.

Recall that the rotation group G is compact, in order to obtain the rotational symmetry of $\bar{u}_f(t, x)$, we only need to check (A4) in view of our abstract Theorem C. This will be done in Propositions 7.3 and 7.5 below. We first proceed to present the following useful lemma.

Lemma 7.2.

$$\sup_{g \in H(f)} \sup_{t \in \mathbb{R}} |\bar{u}_g(t, x)| \rightarrow 0, \quad \text{as } |x| \rightarrow \infty.$$

Proof. Since K is a 1-cover of $H(f)$, for any $g \in H(f)$ there exists a sequence $\{t_n\} \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \bar{u}_{f \cdot t_n}(t, x) = \lim_{n \rightarrow \infty} \bar{u}_{(f \cdot t_n) \cdot t}(0, x) = \bar{u}_{g \cdot t}(0, x) = \bar{u}_g(t, x)$$

uniformly in $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Then for any $\varepsilon > 0$, it follows from (7.2) that there exists some $R_\varepsilon > 0$ such that

$$\begin{aligned} |\bar{u}_g(t, x)| &\leq |\bar{u}_g(t, x) - \bar{u}_{f \cdot t_n}(t, x)| + |\bar{u}_{f \cdot t_n}(t, x)| \\ &= |\bar{u}_g(t, x) - \bar{u}_{f \cdot t_n}(t, x)| + |\bar{u}_f(t + t_n, x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $t \in \mathbb{R}$, $|x| > R_\varepsilon$, $g \in H(f)$ and n sufficiently large. This implies that

$$\sup_{g \in H(f)} \sup_{t \in \mathbb{R}} |\bar{u}_g(t, x)| \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

□

Proposition 7.3. *Let ϵ_0 be given in (f 3). Let also $(u_0, g_0) \in X \times H(f)$ be such that its omega limit set $\mathcal{O}(u_0, g_0)$ exists and satisfies*

$$\|v(\cdot) - \bar{u}_g(0, \cdot)\|_{L^\infty} < \frac{\epsilon_0}{2}, \quad \text{for all } (v, g) \in \mathcal{O}(u_0, g_0),$$

with

$$(v(x), g) \leq (\bar{u}_g(0, x), g), \quad v(x) \not\equiv \bar{u}_g(0, x), \quad x \in \mathbb{R}^n, (v, g) \in \mathcal{O}(u_0, g_0).$$

Then there is a neighborhood $B(e) \subset G$ of e such that

$$(av(x), g) \leq (\bar{u}_g(0, x), g), \quad av(x) \not\equiv \bar{u}_g(0, x),$$

for all $x \in \mathbb{R}^n$, $a \in B(e)$ and $(v, g) \in \mathcal{O}(u_0, g_0)$. The assertion remains true if the inequality sign \leq is replaced by \geq .

Proof. We only prove the first assertion of the Proposition. The last assertion is similar. Motivated by [18, 19, Lemma 5.8], we let α, ϵ_0, R_0 be such that (f 3) holds. By virtue of Lemma 7.2, we choose some $R \geq R_0 > 0$ such that

$$|\bar{u}_g(t, x)| < \frac{\epsilon_0}{4}, \quad \text{for all } x \in \mathbb{R}^n \setminus B_R, \quad g \in H(f) \text{ and } t \in \mathbb{R}, \quad (\mathbf{A})$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Moreover, for such $\epsilon_0 > 0$, there exists a neighborhood $B_0(e) \subset G$ of e such that

$$|a\bar{u}_g(0, x) - \bar{u}_g(0, x)| < \frac{\epsilon_0}{4}, \quad \text{for all } x \in \mathbb{R}^n, a \in B_0(e) \text{ and } g \in H(f). \quad (7.3)$$

Recall that

$$\|v(\cdot) - \bar{u}_g(0, \cdot)\|_{L^\infty} < \frac{\epsilon_0}{2}, \quad \text{for all } (v, g) \in \mathcal{O}(u_0, g_0). \quad (7.4)$$

It then follows from (7.3)-(7.4) and (A) that

$$\begin{aligned}
|av(x)| &\leq |av(x) - a\bar{u}_g(0, x)| + |a\bar{u}_g(0, x)| \\
&\leq |v(a^{-1}x) - \bar{u}_g(0, a^{-1}x)| + |a\bar{u}_g(0, x) - \bar{u}_g(0, x)| + |\bar{u}_g(0, x)| \\
&< \frac{\epsilon_0}{2} + \frac{\epsilon_0}{4} + \frac{\epsilon_0}{4} = \epsilon_0,
\end{aligned}$$

for all $a \in B_0(e)$, $x \in \mathbb{R}^n \setminus B_R$ and $(v, g) \in \mathcal{O}(u_0, g_0)$. That is,

$$|av(x)| < \epsilon_0 \quad \text{for all } a \in B_0(e), x \in \mathbb{R}^n \setminus B_R \text{ and } (v, g) \in \mathcal{O}(u_0, g_0). \quad (\text{B})$$

Noticing that $(v(x), g) \leq (\bar{u}_g(0, x), g)$, $v(x) \not\equiv \bar{u}_g(0, x)$ for $x \in \mathbb{R}^n$ and $(v, g) \in \mathcal{O}(u_0, g_0)$, the strong maximum principle yields that

$$(u(t, x, v, g), g \cdot t) < (\bar{u}_{g \cdot t}(0, x), g \cdot t), \quad \forall x \in \mathbb{R}^n, (v, g) \in \mathcal{O}(u_0, g_0), t > 0.$$

So, by the invariance of $\mathcal{O}(u_0, g_0)$, we obtain that $(v(x), g) < (\bar{u}_g(0, x), g)$, for $x \in \mathbb{R}^n$ and $(v, g) \in \mathcal{O}(u_0, g_0)$. Since $\mathcal{O}(u_0, g_0)$ is compact in $X \times H(f)$, the continuity of $\bar{u}_g(0, \cdot)$ on g implies that there is an $\tilde{\epsilon} > 0$ such that

$$(v(x), g) < (\bar{u}_g(0, x) - \tilde{\epsilon}, g), \quad \text{for all } x \in \overline{B_R} \text{ and } (v, g) \in \mathcal{O}(u_0, g_0).$$

As a consequence, there exists a smaller neighborhood $B(e) \subset B_0(e)$ of e such that

$$(av(x), g) < (\bar{u}_g(0, x), g) \quad \text{for all } a \in B(e), x \in \overline{B_R} \text{ and } (v, g) \in \mathcal{O}(u_0, g_0). \quad (\text{C})$$

Note also that $\bar{u}_g(0, \cdot) - av(\cdot) \geq \bar{u}_g(0, \cdot) - a\bar{u}_g(0, \cdot)$, for all $(v, g) \in \mathcal{O}(u_0, g_0)$. Then one further obtains that

$$\liminf_{|x| \rightarrow \infty} (\bar{u}_g(0, x) - av(x)) \geq \liminf_{|x| \rightarrow \infty} (\bar{u}_g(0, x) - a\bar{u}_g(0, x)) = 0 \quad (\text{D})$$

for all $(v, g) \in \mathcal{O}(u_0, g_0)$ and $a \in B(e)$.

Now we claim that the Proposition follows immediately from (A)-(D). Indeed, for any $(v, g) \in \mathcal{O}(u_0, g_0)$ and $\tau > 0$, one can find some $(v_{-\tau}, g_{-\tau}) \in \mathcal{O}(u_0, g_0)$ such that $\Pi_\tau(v_{-\tau}, g_{-\tau}) = (v, g)$. Then for any $a \in B(e)$, by (A)-(D) and the invariance of $\mathcal{O}(u_0, g_0)$, we have that

- (i) $|\bar{u}_g(t, x)| < \epsilon_0$, for all $x \in \mathbb{R}^n \setminus B_R$, $g \in H(f)$ and $t \in \mathbb{R}$,
- (ii) $|au(t, x, v_{-\tau}, g_{-\tau})| < \epsilon_0$, for all $t > 0$ and $x \in \mathbb{R}^n \setminus B_R$,
- (iii) $au(t, x, v_{-\tau}, g_{-\tau}) < \bar{u}_{g_{-\tau} \cdot t}(0, x)$, for all $t > 0$ and $x \in \partial B_R$,
- (iv) $\liminf_{|x| \rightarrow \infty} (\bar{u}_{g_{-\tau} \cdot t}(0, x) - au(t, x, v_{-\tau}, g_{-\tau})) \geq 0$, for all $t > 0$.

Therefore, Lemma 7.4 below implies that

$$\bar{u}_{g_{-\tau} \cdot t}(0, x) - au(t, x, v_{-\tau}, g_{-\tau}) = \bar{u}_{g_{-\tau} \cdot t}(0, x) - u(t, x, av_{-\tau}, g_{-\tau}) \geq -2\epsilon_0 e^{-\alpha t}$$

for all $x \in \mathbb{R}^n \setminus B_R$ and $t > 0$. In particular (let $t = \tau$),

$$\bar{u}_{g-\tau \cdot \tau}(0, x) - au(\tau, x, v_{-\tau}, g_{-\tau}) \geq -2\epsilon_0 e^{-\alpha\tau}, \quad \text{for all } x \in \mathbb{R}^n \setminus B_R,$$

and hence

$$\bar{u}_g(0, x) - av(x) \geq -2\epsilon_0 e^{-\alpha\tau}, \quad \text{for all } x \in \mathbb{R}^n \setminus B_R.$$

Since $\tau > 0$ is arbitrarily chosen, by letting $\tau \rightarrow \infty$ we have $\bar{u}_g(0, x) \geq av(x)$, for all $x \in \mathbb{R}^n \setminus B_R$, $(v, g) \in \mathcal{O}(u_0, g_0)$ and $a \in B(e)$. Combining with (C), we have completed the proof. \square

Lemma 7.4. *Let α, ϵ_0, R_0 be such that (f 3) holds. Let $R \geq R_0$ be such that*

$$|\bar{u}_g(t, x)| < \epsilon_0, \quad \text{for all } x \in \mathbb{R}^n \setminus B_R, \quad g \in H(f) \text{ and } t \in \mathbb{R}.$$

Let also $u(t, x, v_0, g)$ be a solution of (7.1_g) satisfying

$$|u(t, x, v_0, g)| < \epsilon_0, \quad \forall t > 0, \quad x \in \mathbb{R}^n \setminus B_R.$$

Assume that

$$\bar{u}_g(t, x) \geq u(t, x, v_0, g), \quad \text{for } x \in \partial B_R, \quad t > 0$$

and

$$\liminf_{|x| \rightarrow \infty} (\bar{u}_g(t, x) - u(t, x, v_0, g)) \geq 0, \quad \forall t > 0.$$

Then

$$\bar{u}_g(t, x) - u(t, x, v_0, g) \geq -2\epsilon_0 e^{-\alpha t} \quad \text{for all } x \in \mathbb{R}^n \setminus B_R \text{ and } t > 0.$$

Proof. The proof is similar as [18, Lemma 5.9], we here give the detail for completeness. For any $g \in H(f)$, the function $w(t, x) = \bar{u}_g(t, x) - u(t, x, v_0, g)$ is a solution of the linear parabolic equation

$$\frac{\partial w}{\partial t} = \Delta w + \xi(t, x)w, \quad x \in \mathbb{R}^n \setminus \overline{B_R}, \quad t > 0 \tag{7.5}$$

under the boundary condition $w = \bar{u}_g - u \geq 0$ on ∂B_R , where

$$\xi(t, x) = \int_0^1 g'_u(t, x, \theta \bar{u}_g(t, x) + (1 - \theta)u(t, x, v_0, g)) d\theta.$$

In view of our assumptions, it is easy to see that

$$|\theta \bar{u}_g(t, x) + (1 - \theta)u(t, x, v_0, g)| < \epsilon_0 \quad \text{for all } x \in \mathbb{R}^n \setminus B_R \text{ and } t > 0.$$

Since $g \in H(f)$ satisfies (f 3), we have

$$\xi(t, x) \leq -\alpha \quad \text{for all } x \in \mathbb{R}^n \setminus B_R \text{ and } t > 0.$$

Let $\tilde{r}(t) = -2\epsilon_0 e^{-\alpha t}$. Then

$$\frac{\partial \tilde{r}}{\partial t} \leq \Delta \tilde{r} + \xi(t, x) \tilde{r}, \quad x \in \mathbb{R}^n \setminus \overline{B_R}, \quad t > 0.$$

Clearly, $\tilde{r}(t) < 0 \leq w(t, x)$ on ∂B_R . Moreover,

$$\tilde{r}(0) = -2\epsilon_0 \leq \bar{u}_g(0, x) - v_0(x) = w(0, x), \quad \text{for } x \in \mathbb{R}^n \setminus B_R,$$

and $\tilde{r}(t) < 0 \leq \liminf_{|x| \rightarrow \infty} w(t, x)$ for all $t > 0$. Then it follows from the comparison theorem that

$$\tilde{r}(t) \leq w(t, x) \quad \text{for all } x \in \mathbb{R}^n \setminus B_R \text{ and } t > 0,$$

which completes the proof. \square

Proposition 7.5. *Let ϵ_0 be given in (f 3). Then, for any solution $u(t, x, v_0, g)$ of (7.1_g) satisfying*

$$\sup_{t \geq 0} \|u(t, \cdot, v_0, g) - \bar{u}_g(t, \cdot)\|_{L^\infty} < \frac{\epsilon_0}{4}, \quad (7.6)$$

the forward orbit $O^+(v_0, g)$ is relatively compact in X .

Proof. Since

$$\sup_{t \in \mathbb{R}} |\bar{u}_g(t, x)| \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty, \quad (7.7)$$

let $R > R_0$ be such that $\sup_{t \in \mathbb{R}} |\bar{u}_g(t, x)| \leq \epsilon_*$ for $x \in \mathbb{R}^n \setminus B_R$, where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ and $\epsilon_* = \frac{\epsilon_0}{4}$. In view of (7.6), it yields that

$$|u(t, x, v_0, g)| \leq 2\epsilon_* \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}^n \setminus B_R. \quad (7.8)$$

Furthermore, $u(t, x, v_0, g)$ satisfies the initial boundary value problem

$$\begin{cases} \frac{\partial w}{\partial t} = \Delta w + g(t, x, w), & x \in \mathbb{R}^n \setminus \overline{B_R}, \quad t > 0, \\ w = u, & x \in \partial B_R, \quad t > 0, \\ w(0, x) = v_0(x), & x \in \mathbb{R}^n \setminus B_R. \end{cases} \quad (7.9)$$

Now let ϕ^+ satisfies

$$\begin{cases} \frac{\partial \phi^+}{\partial t} = \Delta \phi^+ - \alpha \phi^+, & x \in \mathbb{R}^n \setminus \overline{B_R}, \quad t > 0, \\ \phi^+ = 3\epsilon_*, & x \in \partial B_R, \quad t > 0, \\ \phi^+(0, x) = 3\epsilon_*, & x \in \mathbb{R}^n \setminus B_R. \end{cases}$$

Then $\hat{u} := \bar{u}_g + \phi^+$ satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + g(t, x, \bar{u}_g) - \alpha \phi^+, & x \in \mathbb{R}^n \setminus \overline{B_R}, \quad t > 0, \\ u = 3\epsilon_* + \bar{u}_g, & x \in \partial B_R, \quad t > 0, \\ u(0, x) = 3\epsilon_* + \bar{u}_g(0, x), & x \in \mathbb{R}^n \setminus B_R. \end{cases}$$

Note that

$$g(t, x, \hat{u}) - g(t, x, \bar{u}_g) + \alpha\phi^+ = \left[\int_0^1 \frac{\partial g}{\partial u}(t, x, \bar{u}_g + \theta\phi^+) d\theta + \alpha \right] \cdot \phi^+. \quad (7.10)$$

Since $|\bar{u}_g(t, x)| \leq \epsilon_*$ and $|\theta\phi^+| \leq |\phi^+| \leq 3\epsilon_*$ on $\mathbb{R}^n \setminus B_R$, one has $|\bar{u}_g + \theta\phi^+| \leq \epsilon_0$. Thus by (f 3) (with f replaced by g), $\int_0^1 \frac{\partial g}{\partial u}(t, x, \bar{u}_g + \theta\phi^+) d\theta \leq -\alpha$. Note also that $\phi^+ > 0$ on $\mathbb{R}^n \setminus \overline{B_R}$. It follows from (7.10) that $g(t, x, \hat{u}) \leq g(t, x, \bar{u}_g) - \alpha\phi^+$, which implies that

$$\begin{cases} \frac{\partial \hat{u}}{\partial t} \geq \Delta \hat{u} + g(t, x, \hat{u}), & x \in \mathbb{R}^n \setminus \overline{B_R}, t > 0, \\ \hat{u} = 3\epsilon_* + \bar{u}_g \geq 2\epsilon_*, & x \in \partial B_R, t > 0, \\ \hat{u}(0, x) = 3\epsilon_* + \bar{u}_g(0, x) \geq 2\epsilon_*, & x \in \mathbb{R}^n \setminus B_R. \end{cases}$$

Combined with (7.8) and (7.9), the comparison principle implies that

$$u(t, x, v_0, g) \leq \bar{u}_g(t, x) + \phi^+(t, x), \quad \forall t \geq 0, x \in \mathbb{R}^n \setminus B_R.$$

Similarly, we can construct ϕ^- satisfying

$$\begin{cases} \frac{\partial \phi^-}{\partial t} = \Delta \phi^- - \alpha\phi^-, & x \in \mathbb{R}^n \setminus \overline{B_R}, t > 0, \\ \phi^- = -3\epsilon_*, & x \in \partial B_R, t > 0, \\ \phi^-(0, x) = -3\epsilon_*, & x \in \mathbb{R}^n \setminus B_R. \end{cases}$$

and obtain that

$$u(t, x, v_0, g) \geq \bar{u}_g(t, x) + \phi^-(t, x), \quad \forall t \geq 0, x \in \mathbb{R}^n \setminus B_R.$$

A direct estimate yields that (see [12, P.94])

$$\lim_{\substack{t \rightarrow +\infty \\ |x| \rightarrow +\infty}} \phi^\pm(t, x) = 0,$$

which implies that

$$\lim_{\substack{t \rightarrow +\infty \\ |x| \rightarrow +\infty}} |u(t, x, v_0, g) - \bar{u}_g(t, x)| = 0. \quad (7.11)$$

In order to prove the relative compactness of $\{u(t, \cdot, v_0, g)\}_{t \in [0, \infty)}$ in X , we note that, by (7.6)-(7.7), $u(t, x, v_0, g)$ is a bounded solution of (7.1_g) in X . Then the standard parabolic estimate shows that $u(t, \cdot, v_0, g)$ is bounded in $C_{loc}^2(\mathbb{R}^n)$. Combining (7.7), (7.11) and the Arzelà-Ascoli Theorem, we obtain the relative compactness of $\{u(t, \cdot, v_0, g)\}_{t \in [0, \infty)}$ in X . \square

7.2 Traveling waves

In this subsection, we will utilize the abstract results in Section 3 to investigate the monotonicity of stable traveling waves for time-almost periodic reaction-diffusion equations with bistable nonlinearities. Our aim is to study such kind of problems from a general point of view. As a simple illustrated example, we consider the following time-almost periodic reaction-diffusion equation of the form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + f(t, u), \quad z \in \mathbb{R}, t > 0, \quad (7.12)$$

where the nonlinearity $f(t, u) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -admissible and uniformly almost periodic in t , real-valued function. Of course, we remark that our approach for (7.12) here can be applicable, with little modification, to monotonicity of stable traveling waves for other various types of equations (see, e.g. [18, 19]) with bistable nonlinearities.

A solution $u(z, t)$ of (7.12) is called an *almost periodic traveling wave* (see, e.g. [27, Section 2.2]), if there are $\phi \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $c \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$u(z, t) = \phi(z - c(t), t),$$

where $\phi(x, t)$ (called *the wave profile*) is almost periodic in t uniformly with respect to x in bounded sets, and $c'(t)$ (called *the wave speed*) is almost periodic in t ; and moreover, the frequency modules

$$\mathcal{M}(\phi(x, \cdot)), \quad \mathcal{M}(c'(\cdot)) \subset \mathcal{M}(f).$$

We restrict our attention to traveling waves satisfying the connecting condition

$$\lim_{x \rightarrow \pm\infty} \phi(x, t) = u_{\pm}^f(t), \quad \text{uniformly for } t \in \mathbb{R},$$

where $u_{\pm}^f(t)$ are spatially homogeneous time-almost periodic solutions of (7.12) with $\mathcal{M}(u_{\pm}^f(\cdot)) \subset \mathcal{M}(f)$. A traveling wave is called a solitary wave if $u_{+}^f(t) = u_{-}^f(t)$ for all $t \in \mathbb{R}$, a traveling front if $u_{-}^f(t) < u_{+}^f(t)$ for all $t \in \mathbb{R}$, or $u_{-}^f(t) > u_{+}^f(t)$ for all $t \in \mathbb{R}$.

In what follows we assume that

(F) there exist an $\epsilon_0 > 0$ and a $\mu > 0$ such that

$$\frac{\partial f}{\partial u}(t, u) \leq -\mu, \quad \text{for } |u - u_{\pm}^f(t)| < \epsilon_0 \text{ and } t \in \mathbb{R}.$$

Let $X = C_{unif}(\mathbb{R})$ denote the space of bounded and uniformly continuous functions on \mathbb{R} endowed with the $L^\infty(\mathbb{R})$ topology. For any $u_0 \in X$, let $u(\cdot, t; u_0, f)$ be the solution of (7.12) with $u(\cdot, 0; u_0, f) = u_0$.

A traveling wave $\phi(z - c(t), t)$ of (7.12) is called *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that, for every $u_0 \in X$, if $s \geq 0$ and $\|u(\cdot, s; u_0, f) -$

$\phi(\cdot - c(s), s)\|_{L^\infty} \leq \delta(\varepsilon)$ then

$$\|u(\cdot, t; u_0, f) - \phi(\cdot - c(t), t)\|_{L^\infty} < \varepsilon \quad \text{for each } t \geq s.$$

Moreover, $\phi(z - c(t), t)$ is called *uniformly stable with asymptotic phase* if it is uniformly stable and there exists a $\delta > 0$ such that if $\|u_0 - \phi(\cdot - c(0), 0)\|_{L^\infty} < \delta$ then

$$\|u(\cdot, t; u_0, f) - \phi(\cdot - c(t) - \sigma, t)\|_{L^\infty} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for some $\sigma \in \mathbb{R}$. A traveling wave $\phi(z - c(t), t)$ is called *spatially monotone* if $\phi(x, t)$ is a non-decreasing or non-increasing function of x for every $t \in \mathbb{R}$.

Based on our main abstract results, Theorems B and D, in Section 3, we derive the following results:

Theorem 7.6. *Any uniformly stable traveling wave of (7.12) is spatially monotone. In particular, solitary waves of (7.12) are not uniformly stable.*

Theorem 7.7. *Any uniformly stable traveling wave of (7.12) is uniformly stable with asymptotic phase.*

Remark 7.8. A converse result to Theorem 7.6, i.e., spatially monotone time-almost periodic traveling waves are uniformly stable, was first obtained by Shen [27]. In [28, 29], she further proved the existence of such traveling wave. The same result as Theorem 7.7 can also be found in Shen [27]. Note that our approach (Theorem D) was introduced in a very general framework, and hence, it can be applied to wider classes of equations with little modification.

Proof of Theorems 7.6 and 7.7. We first rewrite equation (7.12) with the moving coordinate $x = z - c(t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c'(t) \frac{\partial u}{\partial x} + f(t, u), \quad x \in \mathbb{R}, t > 0. \quad (7.13)$$

Obviously, $\phi(z - c(t), t)$ is an almost periodic traveling wave of (7.12) if and only if $\phi(x, t)$ is an almost periodic entire solution of (7.13) satisfying $\mathcal{M}(\phi(x, \cdot)) \subset \mathcal{M}(f)$. In the following, we rewrite $\phi(x, t)$ as $\phi^{y_0}(x, t)$, with $y_0 = (c', f)$, for the sake of completeness. Therefore, it is easy to see that

$$\lim_{x \rightarrow \pm\infty} \phi^{y_0}(x, t) = u_\pm^f(t), \quad \text{uniformly in } t \in \mathbb{R}. \quad (7.14)$$

Let $Y = H(c', f)$ be the hull of the function $y_0 = (c', f)$. By the standard theory of reaction-diffusion systems (see, e.g. [5, 6]), it follows that for every $v_0 \in X$ and $y = (d, g) \in Y$, the system

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + d(t) \frac{\partial u}{\partial x} + g(t, u), \quad x \in \mathbb{R}, t > 0 \quad (7.13_y)$$

admits a (locally) unique regular solution $v(\cdot, t; v_0, y)$ in X with $v(\cdot, 0; v_0, y) = v_0$. This solution also continuously depends on $y \in Y$ and $v_0 \in X$ (see, e.g.

[6, Sec.3.4]). Therefore, (7.13_y) induces a (local) skew-product semiflow Π on $X \times Y$ with

$$\Pi_t(v_0, y) = (v(\cdot, t; v_0, y), y \cdot t), \quad \forall (v_0, y) \in X \times Y, t \geq 0.$$

We define an order relation in X by

$$u \leq v \quad \text{if} \quad u(x) \leq v(x), \quad \forall x \in \mathbb{R}.$$

Let $G = \{a_\sigma : \sigma \in \mathbb{R}\}$ be the group of translations

$$a_\sigma : u(\cdot) \mapsto u(\cdot - \sigma)$$

acting on the space X . Then (A1)-(A3) are fulfilled.

Note that $\phi^{y_0}(x, t)$ is a uniformly almost periodic solution of (7.13) with $\mathcal{M}(\phi^{y_0}(x, \cdot)) \subset \mathcal{M}(f) = \mathcal{M}(y_0)$. So, the closure K of the orbit $\{(\phi^{y_0}(\cdot, t), y_0 \cdot t) : t \in \mathbb{R}\}$ of Π_t is a uniformly stable 1-cover of Y . As a consequence, K can be written as

$$K = \{(\phi^y(\cdot, 0), y) \in X \times Y : y = (d, g) \in Y\},$$

where the map $y \mapsto \phi^y(\cdot, 0) \in X$ is continuous and satisfies $\phi^{y_0}(\cdot, t) = \phi(\cdot, t)$ and $\phi^{y \cdot t}(\cdot, 0) = \phi^y(\cdot, t)$ for all $y \in Y$ and $t \in \mathbb{R}$. By virtue of (7.14), it is not difficult to see that

$$\lim_{x \rightarrow \pm\infty} \phi^y(x, t) = u_\pm^g(t), \quad \text{uniformly for } y = (d, g) \in Y \text{ and } t \in \mathbb{R}, \quad (7.15)$$

where $\{(u_\pm^g(0), g) \in \mathbb{R} \times H(f) : g \in H(f)\}$ is a 1-cover of $H(f)$ and satisfies $u_\pm^{g \cdot t}(0) = u_\pm^g(t)$ for all $g \in H(f)$ and $t \in \mathbb{R}$. Of course, one can also easily see that, for any $g \in H(f)$, the function-pair $(g, u_\pm^g(t))$ also satisfies the condition (F), i.e.,

(F)_g: there exist an $\epsilon_0 > 0$ and a $\mu > 0$ such that

$$\frac{\partial g}{\partial u}(t, u) \leq -\mu, \quad \text{for } |u - u_\pm^g(t)| < \epsilon_0 \text{ and } t \in \mathbb{R}.$$

In order to apply Theorems B and D in Section 3, we have to check (A4) there. By virtue of (7.15) and the condition (F)_g above, (A4-i) can be shown by repeating an analogue of Proposition 7.5, with \bar{u}_g replaced by $\phi^y - u_\pm^g$ (see also the similar arguments in [18, Lemma 5.6]). We omit the detail here.

As for (A4-ii), we will deduce it from Proposition 7.9 below. Based on this, we can apply Theorem B to obtain that the group orbit GK of K is a 1-D subbundle of $X \times Y$. In particular, fix $y_0 \cdot t \in Y$, the fibre

$$GK_{y_0 \cdot t} = G[\phi^{y_0 \cdot t}(x, 0)] = G[\phi^{y_0}(x, t)] = G[\phi(x, t)] = \{\phi(x - \sigma, t) : \sigma \in \mathbb{R}\}$$

is totally-ordered, which implies that $\phi(x, t)$ is monotone in x for every $t \in \mathbb{R}$. Furthermore, it follows from Theorem D that the traveling wave $\phi(z - c(t), t)$ is uniformly stable with asymptotic phase. This completes the proof of Theorems 7.6 and 7.7. \square

Proposition 7.9. *Let ϵ_0 be given in (F). For $(u_0, y_0) \in X \times Y$, suppose that the omega limit set $\mathcal{O}(u_0, y_0)$ exists and satisfies*

$$\|v(\cdot) - \phi^y(\cdot, 0)\|_{L^\infty} < \frac{\epsilon_0}{2} \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0), \quad (7.16)$$

as well as

$$(v(x), y) \leq (\phi^y(x - h, 0), y), \quad v(x) \not\equiv \phi^y(x - h, 0), \quad \forall (v, y) \in \mathcal{O}(u_0, y_0), \quad x \in \mathbb{R}, \quad (7.17)$$

for some $h \in \mathbb{R}$. Then there exists some $\delta > 0$ such that

$$(v(x), y) \leq (\phi^y(x - h - \sigma, 0), y), \quad v(x) \not\equiv \phi^y(x - h - \sigma, 0)$$

for all $(v, y) \in \mathcal{O}(u_0, y_0)$, $x \in \mathbb{R}$ and $|\sigma| < \delta$. The assertion remains true if the inequality sign \leq is replaced by \geq .

Proof. We use the similar arguments in Proposition 7.3. Let μ, ϵ_0 be such that (F) holds. By (7.15), we have

$$\lim_{x \rightarrow \pm\infty} \phi^y(x - h, 0) = \lim_{x \rightarrow \pm\infty} \phi^y(x, 0) = u_\pm^g(0), \quad \text{uniformly for } y = (d, g) \in Y.$$

Thus there exist some $R', R'' > 0$ such that

$$|\phi^y(x, 0) - u_\pm^g(0)| < \frac{\epsilon_0}{2} \quad \text{for all } |x| > R' \text{ and } y \in Y, \quad (7.18)$$

as well as

$$|\phi^y(x - h, 0) - u_\pm^g(0)| < \frac{\epsilon_0}{2} \quad \text{for all } |x| > R'' \text{ and } y \in Y. \quad (7.19)$$

Let $R = \max\{R', R''\}$. In view of (7.16), it follows from (7.18) that

$$|v(x) - u_\pm^g(0)| < \epsilon_0 \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0) \text{ and } |x| > R. \quad (\mathbf{A}')$$

Moreover, combined with (7.19), the continuity of the translation-group action on X implies that there exists a $\delta_0 > 0$ such that if $|\sigma| < \delta_0$ then

$$|\phi^y(x - h - \sigma, 0) - u_\pm^g(0)| < \epsilon_0, \quad \text{for all } |x| > R \text{ and } y \in Y. \quad (\mathbf{B}')$$

Due to the assumption (7.17), the strong maximum principle yields that

$$(v(x, t; v, y), y \cdot t) < (\phi^{y \cdot t}(x - h, 0), y \cdot t), \quad \forall (v, y) \in \mathcal{O}(u_0, y_0), \quad x \in \mathbb{R}, \quad t > 0.$$

By virtue of the invariance of $\mathcal{O}(u_0, y_0)$, we get that

$$(v(x), y) < (\phi^y(x - h, 0), y), \quad \forall (v, y) \in \mathcal{O}(u_0, y_0), \quad x \in \mathbb{R}.$$

Since $\mathcal{O}(u_0, g_0)$ is compact in $X \times Y$, it follows from the continuity of $\phi^y(\cdot, 0)$ on y that for a sufficiently small $\tilde{\epsilon} > 0$,

$$(v(x), y) < (\phi^y(x - h, 0) - \tilde{\epsilon}, y) \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0) \text{ and } |x| \leq R.$$

So one can find a $\delta > 0$ ($\delta \leq \delta_0$) such that if $|\sigma| < \delta$ then

$$(v(x), y) < (\phi^y(x-h-\sigma, 0), y) \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0) \text{ and } |x| \leq R. \quad (\mathbf{C}')$$

Note also that $\phi^y(x-h-\sigma, 0) - v(x) \geq \phi^y(x-h-\sigma, 0) - \phi^y(x-h, 0)$, $\forall (v, y) \in \mathcal{O}(u_0, y_0)$, $x \in \mathbb{R}$. Then

$$\liminf_{|x| \rightarrow \infty} (\phi^y(x-h-\sigma, 0) - v(x)) \geq \liminf_{|x| \rightarrow \infty} (\phi^y(x-h-\sigma, 0) - \phi^y(x-h, 0)) = 0 \quad (\mathbf{D}')$$

for all $(v, y) \in \mathcal{O}(u_0, y_0)$ and $|\sigma| < \delta$.

Similarly as (A)-(D) in the proof of Proposition 7.3, we can deduce from (A')-(D') that, for any $(v, y) \in \mathcal{O}(u_0, y_0)$ and $\tau > 0$, there exists some $(v_{-\tau}, y_{-\tau}) \in \mathcal{O}(u_0, y_0)$ with $\Pi_\tau(v_{-\tau}, y_{-\tau}) = (v, y)$. Moreover, for any $|\sigma| < \delta$, the following statements hold true:

- (i) $|v(x, t; v_{-\tau}, y_{-\tau}) - u_{\pm}^{g-\tau \cdot t}(0)| < \epsilon_0$ for all $t > 0$ and $|x| > R$,
- (ii) $|\phi^y(x-h-\sigma, t) - u_{\pm}^g(t)| < \epsilon_0$ for all $|x| > R$, $y \in Y$ and $t \in \mathbb{R}^+$,
- (iii) $v(x, t; v_{-\tau}, y_{-\tau}) < \phi^{y-\tau \cdot t}(x-h-\sigma, 0)$ for all $t > 0$ and $|x| \leq R$, and
- (iv) $\liminf_{|x| \rightarrow \infty} (\phi^{y-\tau \cdot t}(x-h-\sigma, 0) - v(x, t; v_{-\tau}, y_{-\tau})) \geq 0$ for all $t > 0$.

Therefore, by using an analogue of the last paragraph in the proof of Proposition 7.3 (The proof of this modified version of Lemma 7.4 is almost identical to that of Lemma 7.4), we obtain that

$$\phi^{y-\tau \cdot t}(x-h-\sigma, 0) - v(x, t; v_{-\tau}, y_{-\tau}) \geq -2\epsilon_0 e^{-\mu t} \quad \text{for all } |x| > R \text{ and } t > 0.$$

In particular, by letting $t = \tau$,

$$\phi^y(x-h-\sigma, 0) - v(x) = \phi^{y-\tau \cdot \tau}(x-h-\sigma, 0) - v(x, \tau; v_{-\tau}, y_{-\tau}) \geq -2\epsilon_0 e^{-\mu \tau}, \quad \forall |x| > R.$$

Since $\tau > 0$ is arbitrarily chosen, by letting $\tau \rightarrow \infty$ we have that

$$\phi^y(x-h-\sigma, 0) \geq v(x)$$

for all $|x| > R$, $(v, y) \in \mathcal{O}(u_0, y_0)$ and $|\sigma| < \delta$. Note also (C'). We have proved the Proposition. \square

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